DISCRETE MATHEMATICS

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Chapter 14

GENERATING FUNCTIONS

14.1. Introduction

DEFINITION. By the generating function of a sequence $a_0, a_1, a_2, a_3, \ldots$, we mean the formal power series

$$a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots = \sum_{n=0}^{\infty} a_nX^n.$$

EXAMPLE 14.1.1. The generating function of the sequence

$$\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k}, 0, 0, 0, 0, \ldots$$

is given by

$$\binom{k}{0} + \binom{k}{1}X + \ldots + \binom{k}{k}X^k.$$

This is equal to $(1 + X)^k$ by the Binomial theorem.

EXAMPLE 14.1.2. The generating function of the sequence

$$\underbrace{1, \ldots, 1}_{k}, 0, 0, 0, 0, \ldots$$

is given by

$$1 + X + X^2 + X^3 + \ldots + X^{k-1} = \frac{1 - X^k}{1 - X}.$$

EXAMPLE 14.1.3. The generating function of the sequence $1, 1, 1, 1, \ldots$ is given by

$$1 + X + X^2 + X^3 + \ldots = \sum_{n=0}^{\infty} X^n = \frac{1}{1 - X}.$$

EXAMPLE 14.1.4. The generating function of the sequence $2, 4, 1, 1, 1, 1, \ldots$ is given by

$$\begin{aligned} &2 + 4X + X^2 + X^3 + X^4 + X^5 \ldots = (1 + 3X) + (1 + X + X^2 + X^3 + \ldots) \\ &\quad = (1 + 3X) + \sum_{n=0}^{\infty} X^n = 1 + 3X + \frac{1}{1 - X}. \end{aligned}$$

14.2. Some Simple Observations

The idea used in the Example 14.1.4 can be generalized as follows.

PROPOSITION 14A. *Suppose that the sequences*

$$a_0, a_1, a_2, a_3, \ldots \qquad \textit{and} \qquad b_0, b_1, b_2, b_3, \ldots$$

have generating functions $f(X)$ and $g(X)$ respectively. Then the generating function of the sequence

$$a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$$

is given by $f(X) + g(X)$.

EXAMPLE 14.2.1. The generating function of the sequence $3, 1, 3, 1, 3, 1, \ldots$ can be obtained by combining the generating functions of the two sequences $1, 1, 1, 1, 1, 1, \ldots$ and $2, 0, 2, 0, 2, 0, \ldots$. Now the generating function of the sequence $2, 0, 2, 0, 2, 0, \ldots$ is given by

$$2 + 2X^2 + 2X^4 + \ldots = 2(1 + X^2 + X^4 + \ldots) = \frac{2}{1 - X^2}.$$

It follows that the generating function of the sequence $3, 1, 3, 1, 3, 1, \ldots$ is given by

$$\frac{1}{1 - X} + \frac{2}{1 - X^2}.$$

Sometimes, differentiation and integration of formal power series can be used to obtain the generating functions of various sequences.

EXAMPLE 14.2.2. The generating function of the sequence $1, 2, 3, 4, \ldots$ is given by

$$\begin{aligned} &1 + 2X + 3X^2 + 4X^3 + \ldots = \frac{\mathrm{d}}{\mathrm{d}X}(X + X^2 + X^3 + X^4 + \ldots) \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}X}(1 + X + X^2 + X^3 + X^4 + \ldots) = \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{1}{1 - X}\right) = \frac{1}{(1 - X)^2}. \end{aligned}$$

EXAMPLE 14.2.3. The generating function of the sequence

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

is given by

$$X + \frac{X^2}{2} + \frac{X^3}{3} + \frac{X^4}{4} + \ldots = \int (1 + X + X^2 + X^3 + \ldots)\, \mathrm{d}X = \int \left(\frac{1}{1-X}\right) \mathrm{d}X = C - \log(1-X),$$

where C is an absolute constant. The special case $X = 0$ gives $C = 0$, so that

$$X + \frac{X^2}{2} + \frac{X^3}{3} + \frac{X^4}{4} - \ldots = -\log(1-X).$$

Another simple observation is the following.

PROPOSITION 14B. *Suppose that the sequence $a_0, a_1, a_2, a_3, \ldots$ has generating function $f(X)$. Then for every $k \in \mathbb{N}$, the generating function of the (delayed) sequence*

(1) $$\underbrace{0, \ldots, 0}_{k}, a_0, a_1, a_2, a_3, \ldots$$

is given by $X^k f(X)$.

PROOF. Note that the generating function of the sequence (1) is given by

$$a_0X^k + a_1X^{k+1} + a^2X^{K+2} + a_3X^{K+3} + \ldots = X^k(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)$$

as required. ○

EXAMPLE 14.2.4. The generating function of the sequence $0, 1, 2, 3, 4, \ldots$ is given by $X/(1-X)^2$, and the generating function of the sequence $0, 0, 0, 0, 0, 1, 2, 3, 4, \ldots$ is given by $X^5/(1-X)^2$. On the other hand, the generating function of the sequence $0, 0, 0, 0, 0, 0, 0, 3, 1, 3, 1, 3, 1, \ldots$ is given by

$$\frac{X^7}{1-X} + \frac{2X^7}{1-X^2}.$$

EXAMPLE 14.2.5. Consider the sequence $a_0, a_1, a_2, a_3, \ldots$ where $a_n = n^2 + n$ for every $n \in \mathbb{N} \cup \{0\}$. To find the generating function of this sequence, let $f(X)$ and $g(X)$ denote respectively the generating functions of the sequences

$$0, 1^2, 2^2, 3^2, 4^2, \ldots \qquad \text{and} \qquad 0, 1, 2, 3, 4, \ldots.$$

Note from the Example 14.2.4 that

$$g(X) = \frac{X}{(1-X)^2}.$$

To find $f(X)$, note that the generating function of the sequence $1^2, 2^2, 3^2, 4^2, \ldots$ is given by

$$1^2 + 2^2X + 3^2X^2 + 4^2X^3 + \ldots = \frac{\mathrm{d}}{\mathrm{d}X}(X + 2X^2 + 3X^3 + 4X^4 + \ldots)$$
$$= \frac{\mathrm{d}}{\mathrm{d}X}(g(X)) = \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{X}{(1-X)^2}\right) = \frac{1+X}{(1-X)^3}.$$

It follows from Proposition 14B that

$$f(X) = \frac{X(1+X)}{(1-X)^3}.$$

Finally, in view of Proposition 14A, the required generating function is given by

$$f(X) + g(X) = \frac{X(1+X)}{(1-X)^3} + \frac{X}{(1-X)^2} = \frac{2X}{(1-X)^3}.$$

14.3. The Extended Binomial Theorem

When we use generating function techniques to study problems in discrete mathematics, we very often need to study expressions like

$$(1+Y)^{-k},$$

where $k \in \mathbb{N}$. We can write down a formal power series expansion for the function as follows.

PROPOSITION 14C. (EXTENDED BINOMIAL THEOREM) *Suppose that $k \in \mathbb{N}$. Formally we have*

$$(1+Y)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} Y^n,$$

where for every $n = 0, 1, 2, \ldots$, the extended binomial coefficient is given by

$$\binom{-k}{n} = \frac{-k(-k-1)\ldots(-k-n+1)}{n!}.$$

PROPOSITION 14D. *Suppose that $k \in \mathbb{N}$. Then for every $n = 0, 1, 2, \ldots$, we have*

$$\binom{-k}{n} = (-1)^n \binom{n+k-1}{n}.$$

PROOF. We have

$$\begin{aligned}\binom{-k}{n} &= \frac{-k(-k-1)\ldots(-k-n+1)}{n!} = (-1)^n \frac{k(k+1)\ldots(k+n-1)}{n!} \\ &= (-1)^n \frac{(n+k-1)(n+k-2)\ldots(n+k-1-n+1)}{n!} = (-1)^n \binom{n+k-1}{n}\end{aligned}$$

as required. ○

EXAMPLE 14.3.1. We have

$$(1-X)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} (-X)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} X^n,$$

so that the coefficient of X^n in $(1-X)^{-k}$ is given by

$$(-1)^n \binom{-k}{n} = \binom{n+k-1}{n}.$$

EXAMPLE 14.3.2. We have

$$(1+2X)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} (2X)^n = \sum_{n=0}^{\infty} 2^n \binom{-k}{n} X^n,$$

so that the coefficient of X^n in $(1+2X)^{-k}$ is given by

$$2^n \binom{-k}{n} = (-2)^n \binom{n+k-1}{n}.$$

PROBLEMS FOR CHAPTER 14

1. Find the generating function for each of the following sequences:
   a) $1, 3, 9, 27, 81, \ldots$
   b) $0, 0, 0, 0, 2, -2, 2, -2, 2, -2, \ldots$
   c) $0, 0, 0, 0, 3, -2, 3, -2, 3, -2, \ldots$
   d) $2, 4, 6, 8, 10, \ldots$
   e) $3, 5, 7, 9, 11, \ldots$
   f) $0, 2, 3, 4, 5, 6, \ldots$
   g) $2, -3, 1, 1, 1, 1, 1, \ldots$
   h) $0, 1, 5, 25, 125, 625, \ldots$
   i) $1, \pi, 3, 4, 5, 6, 7, \ldots$
   j) $1, 1, 1, 0, 1, 1, 1, 1, 1, \ldots$

2. Find the generating function for the sequence $a_n = 3n^2 + 7n + 1$ for $n \in \mathbb{N} \cup \{0\}$.

3. Find the generating function of the sequence

$$0, 0, 0, 1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \ldots,$$

   explaining carefully every step in your argument.

4. Find the generating function of the sequence $a_0, a_1, a_2, \ldots$, where

$$a_n = 3^{n+1} + 2n + \binom{3}{n} + 4\binom{-5}{n}.$$

5. Find the first four terms of the formal power series expansion of $(1-3X)^{-13}$.