

# DISCRETE MATHEMATICS

W W L CHEN

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## Chapter 15

### NUMBER OF SOLUTIONS OF A LINEAR EQUATION

#### 15.1. Introduction

EXAMPLE 15.1.1. Suppose that 5 new academic positions are to be awarded to 4 departments in the university, with the restriction that no department is to be awarded more than 3 such positions, and that the Mathematics Department is to be awarded at least 1. We would like to find out in how many ways this could be achieved. If we denote the departments by  $M, P, C, E$ , where  $M$  denotes the Mathematics Department, and denote by  $u_M, u_P, u_C, u_E$  the number of positions awarded to these departments respectively. Then clearly we must have

$$(1) \quad u_M + u_P + u_C + u_E = 5.$$

Furthermore,

$$(2) \quad u_M \in \{1, 2, 3\} \quad \text{and} \quad u_P, u_C, u_E \in \{0, 1, 2, 3\}.$$

We therefore need to find the number of solutions of the equation (1), subject to the restriction (2).

In general, we would like to find the number of solutions of an equation of the type

$$u_1 + \dots + u_k = n,$$

where  $n, k \in \mathbb{N}$  are given, and where the variables  $u_1, \dots, u_k$  are to assume integer values, subject to certain given restrictions.

**15.2. Case A – The Simplest Case**

Suppose that we are interested in finding the number of solutions of an equation of the type

$$(3) \quad u_1 + \dots + u_k = n,$$

where  $n, k \in \mathbb{N}$  are given, and where the variables

$$(4) \quad u_1, \dots, u_k \in \{0, 1, 2, 3, \dots\}.$$

**PROPOSITION 15A.** *The number of solutions of the equation (3), subject to the restriction (4), is given by the binomial coefficient*

$$\binom{n+k-1}{n}.$$

PROOF. Consider a row of  $(n+k-1) +$  signs as shown in the picture below:

$$\underbrace{+++++ \dots +++++}_{n+k-1}$$

Let us choose  $n$  of these  $+$  signs and change them to 1's. Clearly there are exactly  $(k-1) +$  signs remaining, the same number of  $+$  signs as in the equation (3). The new situation is shown in the picture below:

$$(5) \quad \underbrace{\underbrace{1 \dots 1}_{u_1} + \dots + \underbrace{1 \dots 1}_{u_k}}_k$$

For example, the picture

$$+1111 + +1 + 111 + 11 + \dots + 1111111 + 1111 + 11$$

denotes the information

$$0 + 4 + 0 + 1 + 3 + 2 + \dots + 7 + 4 + 2$$

(note that consecutive  $+$  signs indicate an empty block of 1's in between, and the  $+$  sign at the left-hand end indicates an empty block of 1's at the left-hand end; similarly a  $+$  sign at the right-hand end would indicate an empty block of 1's at the right-hand end). It follows that our choice of the  $n$  1's corresponds to a solution of the equation (3), subject to the restriction (4). Conversely, any solution of the equation (3), subject to the restriction (4), can be illustrated by a picture of the type (5), and so corresponds to a choice of the  $n$  1's. Hence the number of solutions of the equation (3), subject to the restriction (4), is equal to the number of ways we can choose  $n$  objects out of  $(n+k-1)$ . Clearly this is given by the binomial coefficient indicated.  $\circ$

EXAMPLE 15.2.1. The equation  $u_1 + \dots + u_4 = 11$ , where the variables  $u_1, \dots, u_4 \in \{0, 1, 2, 3, \dots\}$ , has

$$\binom{11+4-1}{11} = \binom{14}{11} = 364$$

solutions.

### 15.3. Case B – Inclusion-Exclusion

We next consider the situation when the variables have upper restrictions. Suppose that we are interested in finding the number of solutions of an equation of the type

$$(6) \quad u_1 + \dots + u_k = n,$$

where  $n, k \in \mathbb{N}$  are given, and where the variables

$$(7) \quad u_1 \in \{0, 1, \dots, m_1\}, \quad \dots, \quad u_k \in \{0, 1, \dots, m_k\}.$$

Our approach is to first of all relax the upper restrictions in (7), and solve instead the Case A problem of finding the number of solutions of the equation (6), where the variables

$$(8) \quad u_1, \dots, u_k \in \{0, 1, 2, 3, \dots\}.$$

By Proposition 15A, this relaxed system has

$$\binom{n+k-1}{n}$$

solutions. Among these solutions will be some which violate the upper restrictions on the variables  $u_1, \dots, u_k$  as given in (7).

For every  $i = 1, \dots, k$ , let  $S_i$  denote the collection of those solutions of (6), subject to the relaxed restriction (8) but which  $u_i > m_i$ . Then we need to calculate

$$|S_1 \cup \dots \cup S_k|,$$

the number of solutions which violate the upper restrictions on the variables  $u_1, \dots, u_k$  as given in (7). This can be achieved by using the Inclusion-exclusion principle, but we need to calculate

$$|S_{i_1} \cap \dots \cap S_{i_j}|$$

whenever  $1 \leq i_1 < \dots < i_j \leq k$ . To do this, we need the following simple observation.

**PROPOSITION 15B.** *Suppose that  $i = 1, \dots, k$  is chosen and fixed. Then the number of solutions of the equation (6), subject to the restrictions*

$$(9) \quad u_i \in \{m_i + 1, m_i + 2, m_i + 3, \dots\}$$

and

$$(10) \quad u_1 \in \mathcal{B}_1, \quad \dots, \quad u_{i-1} \in \mathcal{B}_{i-1}, \quad u_{i+1} \in \mathcal{B}_{i+1}, \quad \dots, \quad u_k \in \mathcal{B}_k,$$

where  $\mathcal{B}_1, \dots, \mathcal{B}_{i-1}, \mathcal{B}_{i+1}, \dots, \mathcal{B}_k$  are all subsets of  $\mathbb{N} \cup \{0\}$ , is equal to the number of solutions of the equation

$$(11) \quad u_1 + \dots + u_{i-1} + v_i + u_{i+1} + \dots + u_k = n - (m_i + 1),$$

subject to the restrictions

$$(12) \quad v_i \in \{0, 1, 2, 3, \dots\}$$

and (10).

PROOF. This is immediate on noting that if we write

$$u_i = v_i + (m_i + 1),$$

then the restrictions (9) and (12) are the same. Furthermore, equation (6) becomes

$$u_1 + \dots + u_{i-1} + (v_i + (m_i + 1)) + u_{i+1} + \dots + u_k = n,$$

which is the same as equation (11).  $\circ$

By applying Proposition 15B successively on the subscripts  $i_1, \dots, i_j$ , we have the following result.

**PROPOSITION 15C.** *Suppose that  $1 \leq i_1 < \dots < i_j \leq k$ . Then*

$$|S_{i_1} \cap \dots \cap S_{i_j}| = \binom{n - (m_{i_1} + 1) - \dots - (m_{i_j} + 1) + k - 1}{n - (m_{i_1} + 1) - \dots - (m_{i_j} + 1)}.$$

PROOF. Note that  $|S_{i_1} \cap \dots \cap S_{i_j}|$  is the number of solutions of the equation (6), subject to the restrictions

$$(13) \quad u_i \in \{m_i + 1, m_i + 2, m_i + 3, \dots\} \quad \text{whenever } i \in \{i_1, \dots, i_j\},$$

and

$$(14) \quad u_i \in \{0, 1, 2, 3, \dots\} \quad \text{whenever } i \notin \{i_1, \dots, i_j\}.$$

Now write

$$u_i = \begin{cases} v_i + (m_i + 1) & \text{if } i \in \{i_1, \dots, i_j\}, \\ v_i & \text{if } i \notin \{i_1, \dots, i_j\}. \end{cases}$$

Then by repeated application of Proposition 15B, we can show that the number of solutions of the equation (6), subject to the restrictions (13) and (14), is equal to the number of solutions of the equation

$$(15) \quad v_1 + \dots + v_k = n - (m_{i_1} + 1) - \dots - (m_{i_j} + 1),$$

subject to the restriction

$$(16) \quad v_i \in \{0, 1, 2, 3, \dots\}.$$

This is a Case A problem, and it follows from Proposition 15A that the number of solutions of the equation (15), subject to the restriction (16), is given by

$$\binom{n - (m_{i_1} + 1) - \dots - (m_{i_j} + 1) + k - 1}{n - (m_{i_1} + 1) - \dots - (m_{i_j} + 1)}.$$

This completes the proof.  $\circ$

EXAMPLE 15.3.1. Consider the equation

$$(17) \quad u_1 + \dots + u_4 = 11,$$

where the variables

$$(18) \quad u_1, \dots, u_4 \in \{0, 1, 2, 3\}.$$

Recall that the equation (17), subject to the restriction

$$(19) \quad u_1, \dots, u_4 \in \{0, 1, 2, 3, \dots\},$$

has

$$\binom{11+4-1}{11} = \binom{14}{11}$$

solutions. For every  $i = 1, \dots, 4$ , let  $S_i$  denote the collection of those solutions of (17), subject to the relaxed restriction (19) but which  $u_i > 3$ . Then we need to calculate

$$|S_1 \cup \dots \cup S_4|.$$

Note that by Proposition 15C,

$$|S_1| = |S_2| = |S_3| = |S_4| = \binom{11-4+4-1}{11-4} = \binom{10}{7}$$

and

$$|S_1 \cap S_2| = |S_1 \cap S_3| = |S_1 \cap S_4| = |S_2 \cap S_3| = |S_2 \cap S_4| = |S_3 \cap S_4| = \binom{11-8+4-1}{11-8} = \binom{6}{3}.$$

Next, note that a similar argument gives

$$|S_1 \cap S_2 \cap S_3| = \binom{11-12+4-1}{11-12} = \binom{2}{-1}.$$

This is meaningless. However, note that  $|S_1 \cap S_2 \cap S_3|$  is the number of solutions of the equation

$$(v_1 + 4) + (v_2 + 4) + (v_3 + 4) + v_4 = 11,$$

subject to the restriction

$$v_1, v_2, v_3, v_4 \in \{0, 1, 2, 3, \dots\}.$$

This clearly has no solution. Hence

$$|S_1 \cap S_2 \cap S_3| = |S_1 \cap S_2 \cap S_4| = |S_1 \cap S_3 \cap S_4| = |S_2 \cap S_3 \cap S_4| = 0.$$

Similarly

$$|S_1 \cap S_2 \cap S_3 \cap S_4| = 0.$$

It then follows from the Inclusion-exclusion principle that

$$\begin{aligned} |S_1 \cup S_2 \cup S_3 \cup S_4| &= \sum_{j=1}^4 (-1)^{j+1} \sum_{1 \leq i_1 < \dots < i_j \leq 4} |S_{i_1} \cap \dots \cap S_{i_j}| \\ &= (|S_1| + |S_2| + |S_3| + |S_4|) \\ &\quad - (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|) \\ &\quad + (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|) \\ &\quad - (|S_1 \cap S_2 \cap S_3 \cap S_4|) \\ &= \binom{4}{1} \binom{10}{7} - \binom{4}{2} \binom{6}{3}. \end{aligned}$$

It now follows that the number of solutions of the equation (17), subject to the restriction (18), is given by

$$\binom{14}{11} - |S_1 \cup S_2 \cup S_3 \cup S_4| = \binom{14}{11} - \binom{4}{1} \binom{10}{7} + \binom{4}{2} \binom{6}{3} = 4.$$

#### 15.4. Case C – A Minor Irritation

We next consider the situation when the variables have non-standard lower restrictions. Suppose that we are interested in finding the number of solutions of an equation of the type

$$(20) \quad u_1 + \dots + u_k = n,$$

where  $n, k \in \mathbb{N}$  are given, and where for each  $i = 1, \dots, k$ , the variable

$$(21) \quad u_i \in \mathcal{I}_i,$$

where

$$(22) \quad \mathcal{I}_i = \{p_i, \dots, m_i\} \quad \text{or} \quad \mathcal{I}_i = \{p_i, p_i + 1, p_i + 2, \dots\}.$$

It turns out that we can apply the same idea as in Proposition 15B to reduce the problem to a Case A problem or a Case B problem. For every  $i = 1, \dots, k$ , write  $u_i = v_i + p_i$ , and let  $\mathcal{J}_i = \{x - p_i : x \in \mathcal{I}_i\}$ ; in other words, the set  $\mathcal{J}_i$  is obtained from the set  $\mathcal{I}_i$  by subtracting  $p_i$  from every element of  $\mathcal{I}_i$ . Clearly

$$(23) \quad \mathcal{J}_i = \{0, \dots, m_i - p_i\} \quad \text{or} \quad \mathcal{J}_i = \{0, 1, 2, \dots\}.$$

Note also that the equation (20) becomes

$$(24) \quad v_1 + \dots + v_k = n - p_1 - \dots - p_k.$$

It is easy to see that the number of solutions of the equation (20), subject to the restrictions (21) and (22), is the same as the number of solutions of the equation (24), subject to the restrictions  $v_i \in \mathcal{J}_i$  and (23).

EXAMPLE 15.4.1. Consider the equation

$$(25) \quad u_1 + \dots + u_4 = 11,$$

where the variables

$$(26) \quad u_1 \in \{1, 2, 3\} \quad \text{and} \quad u_2, u_3, u_4 \in \{0, 1, 2, 3\}.$$

We can write  $u_1 = v_1 + 1$ ,  $u_2 = v_2$ ,  $u_3 = v_3$  and  $u_4 = v_4$ . Then

$$(27) \quad v_1 \in \{0, 1, 2\} \quad \text{and} \quad v_2, v_3, v_4 \in \{0, 1, 2, 3\}.$$

Furthermore, the equation (25) becomes

$$(28) \quad v_1 + \dots + v_4 = 10.$$

The number of solutions of the equation (25), subject to the restriction (26), is equal to the number of solutions of the equation (28), subject to the restriction (27). We therefore have a Case B problem.

EXAMPLE 15.4.2. Consider the equation

$$(29) \quad u_1 + \dots + u_6 = 23,$$

where the variables

$$(30) \quad u_1, u_2, u_3 \in \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad u_4, u_5 \in \{2, 3, 4, 5\} \quad \text{and} \quad u_6 \in \{3, 4, 5\}.$$

We can write  $u_1 = v_1 + 1$ ,  $u_2 = v_2 + 1$ ,  $u_3 = v_3 + 1$ ,  $u_4 = v_4 + 2$ ,  $u_5 = v_5 + 2$  and  $u_6 = v_6 + 3$ . Then

$$(31) \quad v_1, v_2, v_3 \in \{0, 1, 2, 3, 4, 5\} \quad \text{and} \quad v_4, v_5 \in \{0, 1, 2, 3\} \quad \text{and} \quad v_6 \in \{0, 1, 2\}.$$

Furthermore, the equation (29) becomes

$$(32) \quad v_1 + \dots + v_6 = 13.$$

The number of solutions of the equation (29), subject to the restriction (30), is equal to the number of solutions of the equation (32), subject to the restriction (31). We therefore have a Case B problem. Consider first of all the Case A problem, where the upper restrictions on the variables  $v_1, \dots, v_6$  are relaxed. By Proposition 15A, the number of solutions of the equation (32), subject to the relaxed restriction

$$(33) \quad v_1, \dots, v_6 \in \{0, 1, 2, 3, \dots\},$$

is given by the binomial coefficient

$$\binom{13 + 6 - 1}{13} = \binom{18}{13}.$$

Let

$$\begin{aligned} S_1 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_1 > 5\}, \\ S_2 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_2 > 5\}, \\ S_3 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_3 > 5\}, \\ S_4 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_4 > 3\}, \\ S_5 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_5 > 3\}, \\ S_6 &= \{(v_1, \dots, v_6) : (32) \text{ and } (33) \text{ hold and } v_6 > 2\}. \end{aligned}$$

Then we need to calculate

$$|S_1 \cup \dots \cup S_6|.$$

Note that by Proposition 15C,

$$\begin{aligned} |S_1| &= |S_2| = |S_3| = \binom{13 - 6 + 6 - 1}{13 - 6} = \binom{12}{7}, \\ |S_4| &= |S_5| = \binom{13 - 4 + 6 - 1}{13 - 4} = \binom{14}{9}, \\ |S_6| &= \binom{13 - 3 + 6 - 1}{13 - 3} = \binom{15}{10}. \end{aligned}$$

Also

$$\begin{aligned}
|S_1 \cap S_2| &= |S_1 \cap S_3| = |S_2 \cap S_3| = \binom{13-12+6-1}{13-12} = \binom{6}{1}, \\
|S_1 \cap S_4| &= |S_1 \cap S_5| = |S_2 \cap S_4| = |S_2 \cap S_5| = |S_3 \cap S_4| = |S_3 \cap S_5| = \binom{13-10+6-1}{13-10} = \binom{8}{3}, \\
|S_1 \cap S_6| &= |S_2 \cap S_6| = |S_3 \cap S_6| = \binom{13-9+6-1}{13-9} = \binom{9}{4}, \\
|S_4 \cap S_5| &= \binom{13-8+6-1}{13-8} = \binom{10}{5}, \\
|S_4 \cap S_6| &= |S_5 \cap S_6| = \binom{13-7+6-1}{13-7} = \binom{11}{6},
\end{aligned}$$

and

$$\begin{aligned}
|S_1 \cap S_2 \cap S_3| &= |S_1 \cap S_2 \cap S_4| = |S_1 \cap S_2 \cap S_5| = |S_1 \cap S_2 \cap S_6| = |S_1 \cap S_3 \cap S_4| \\
&= |S_1 \cap S_3 \cap S_5| = |S_1 \cap S_3 \cap S_6| = |S_1 \cap S_4 \cap S_5| = |S_2 \cap S_3 \cap S_4| \\
&= |S_2 \cap S_3 \cap S_5| = |S_2 \cap S_3 \cap S_6| = |S_2 \cap S_4 \cap S_5| = |S_3 \cap S_4 \cap S_5| = 0, \\
|S_1 \cap S_4 \cap S_6| &= |S_1 \cap S_5 \cap S_6| = |S_2 \cap S_4 \cap S_6| = |S_2 \cap S_5 \cap S_6| = |S_3 \cap S_4 \cap S_6| \\
&= |S_3 \cap S_5 \cap S_6| = \binom{13-13+6-1}{13-13} = \binom{5}{0}, \\
|S_4 \cap S_5 \cap S_6| &= \binom{13-11+6-1}{13-11} = \binom{7}{2}.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
|S_{i_1} \cap \dots \cap S_{i_4}| &= 0 && \text{whenever } 1 \leq i_1 < \dots < i_4 \leq 6, \\
|S_{i_1} \cap \dots \cap S_{i_5}| &= 0 && \text{whenever } 1 \leq i_1 < \dots < i_5 \leq 6,
\end{aligned}$$

and

$$|S_1 \cap \dots \cap S_6| = 0.$$

It follows from the Inclusion-exclusion principle that

$$\begin{aligned}
|S_1 \cup \dots \cup S_6| &= \sum_{j=1}^6 (-1)^{j+1} \sum_{1 \leq i_1 < \dots < i_j \leq 6} |S_{i_1} \cap \dots \cap S_{i_j}| \\
&= \left( 3 \binom{12}{7} + 2 \binom{14}{9} + \binom{15}{10} \right) - \left( 3 \binom{6}{1} + 6 \binom{8}{3} + 3 \binom{9}{4} + \binom{10}{5} + 2 \binom{11}{6} \right) + \left( 6 \binom{5}{0} + \binom{7}{2} \right).
\end{aligned}$$

Hence the number of solutions of the equation (29), subject to the restriction (30), is equal to

$$\begin{aligned}
&\binom{18}{13} - |S_1 \cup \dots \cup S_6| \\
&= \binom{18}{13} - \left( 3 \binom{12}{7} + 2 \binom{14}{9} + \binom{15}{10} \right) \\
&\quad + \left( 3 \binom{6}{1} + 6 \binom{8}{3} + 3 \binom{9}{4} + \binom{10}{5} + 2 \binom{11}{6} \right) - \left( 6 \binom{5}{0} + \binom{7}{2} \right).
\end{aligned}$$

### 15.5. Case Z – A Major Irritation

Suppose that we are interested in finding the number of solutions of the equation

$$u_1 + \dots + u_4 = 23,$$

where the variables

$$u_1, u_2 \in \{1, 3, 5, 7, 9\} \quad \text{and} \quad u_3, u_4 \in \{3, 6, 7, 8, 9, 10, 11, 12\}.$$

Then it is very difficult and complicated, though possible, to make our methods discussed earlier work. We therefore need a slightly better approach. We turn to generating functions which we first studied in Chapter 14.

### 15.6. The Generating Function Method

Suppose that we are interested in finding the number of solutions of an equation of the type

$$(34) \quad u_1 + \dots + u_k = n,$$

where  $n, k \in \mathbb{N}$  are given, and where, for every  $i = 1, \dots, k$ , the variable

$$(35) \quad u_i \in \mathcal{B}_i,$$

where

$$(36) \quad \mathcal{B}_1, \dots, \mathcal{B}_k \subseteq \mathbb{N} \cup \{0\}.$$

For every  $i = 1, \dots, k$ , consider the formal (possibly finite) power series

$$(37) \quad f_i(X) = \sum_{u_i \in \mathcal{B}_i} X^{u_i}$$

corresponding to the variable  $u_i$  in the equation (34), and consider the product

$$f(X) = f_1(X) \dots f_k(X) = \left( \sum_{u_1 \in \mathcal{B}_1} X^{u_1} \right) \dots \left( \sum_{u_k \in \mathcal{B}_k} X^{u_k} \right) = \sum_{u_1 \in \mathcal{B}_1} \dots \sum_{u_k \in \mathcal{B}_k} X^{u_1 + \dots + u_k}.$$

We now rearrange the right-hand side and sum over all those terms for which  $u_1 + \dots + u_k = n$ . Then we have

$$f(X) = \sum_{n=0}^{\infty} \left( \sum_{\substack{u_1 \in \mathcal{B}_1, \dots, u_k \in \mathcal{B}_k \\ u_1 + \dots + u_k = n}} X^{u_1 + \dots + u_k} \right) = \sum_{n=0}^{\infty} \left( \sum_{\substack{u_1 \in \mathcal{B}_1, \dots, u_k \in \mathcal{B}_k \\ u_1 + \dots + u_k = n}} 1 \right) X^n.$$

We have therefore proved the following important result.

**PROPOSITION 15D.** *The number of solutions of the equation (34), subject to the restrictions (35) and (36), is given by the coefficient of  $X^n$  in the formal (possibly finite) power series*

$$f(X) = f_1(X) \dots f_k(X),$$

where, for each  $i = 1, \dots, k$ , the formal (possibly finite) power series  $f_i(X)$  is given by (37).

The main difficulty in this method is to calculate this coefficient.

EXAMPLE 15.6.1. The number of solutions of the equation (29), subject to the restriction (30), is given by the coefficient of  $X^{23}$  in the formal power series

$$\begin{aligned} f(X) &= (X^1 + \dots + X^6)^3 (X^2 + \dots + X^5)^2 (X^3 + \dots + X^5) \\ &= \left( \frac{X - X^7}{1 - X} \right)^3 \left( \frac{X^2 - X^6}{1 - X} \right)^2 \left( \frac{X^3 - X^6}{1 - X} \right) \\ &= X^{10} (1 - X^6)^3 (1 - X^4)^2 (1 - X^3) (1 - X)^{-6}; \end{aligned}$$

or the coefficient of  $X^{13}$  in the formal power series

$$g(X) = (1 - X^6)^3 (1 - X^4)^2 (1 - X^3) (1 - X)^{-6} = h(X) (1 - X)^{-6},$$

where

$$\begin{aligned} h(X) &= (1 - X^6)^3 (1 - X^4)^2 (1 - X^3) \\ &= (1 - 3X^6 + 3X^{12} - X^{18})(1 - 2X^4 + X^8)(1 - X^3) \\ &= 1 - X^3 - 2X^4 - 3X^6 + 2X^7 + X^8 + 3X^9 + 6X^{10} - X^{11} + 3X^{12} - 6X^{13} \\ &\quad + \text{terms with higher powers in } X. \end{aligned}$$

It follows that the coefficient of  $X^{13}$  in  $g(X)$  is given by

$$\begin{aligned} &1 \times \text{coefficient of } X^{13} \text{ in } (1 - X)^{-6} - 1 \times \text{coefficient of } X^{10} \text{ in } (1 - X)^{-6} \\ &- 2 \times \text{coefficient of } X^9 \text{ in } (1 - X)^{-6} - 3 \times \text{coefficient of } X^7 \text{ in } (1 - X)^{-6} \\ &+ 2 \times \text{coefficient of } X^6 \text{ in } (1 - X)^{-6} + 1 \times \text{coefficient of } X^5 \text{ in } (1 - X)^{-6} \\ &+ 3 \times \text{coefficient of } X^4 \text{ in } (1 - X)^{-6} + 6 \times \text{coefficient of } X^3 \text{ in } (1 - X)^{-6} \\ &- 1 \times \text{coefficient of } X^2 \text{ in } (1 - X)^{-6} + 3 \times \text{coefficient of } X^1 \text{ in } (1 - X)^{-6} \\ &- 6 \times \text{coefficient of } X^0 \text{ in } (1 - X)^{-6}. \end{aligned}$$

Using Example 14.3.1, we see that this is equal to

$$\begin{aligned} &(-1)^{13} \binom{-6}{13} - (-1)^{10} \binom{-6}{10} - 2(-1)^9 \binom{-6}{9} - 3(-1)^7 \binom{-6}{7} + 2(-1)^6 \binom{-6}{6} + (-1)^5 \binom{-6}{5} \\ &\quad + 3(-1)^4 \binom{-6}{4} + 6(-1)^3 \binom{-6}{3} - (-1)^2 \binom{-6}{2} + 3(-1)^1 \binom{-6}{1} - 6(-1)^0 \binom{-6}{0} \\ &= \binom{18}{13} - \binom{15}{10} - 2 \binom{14}{9} - 3 \binom{12}{7} + 2 \binom{11}{6} + \binom{10}{5} + 3 \binom{9}{4} + 6 \binom{8}{3} - \binom{7}{2} + 3 \binom{6}{1} - 6 \binom{5}{0} \end{aligned}$$

as in Example 15.4.2.

EXAMPLE 15.6.2. The number of solutions of the equation

$$u_1 + \dots + u_{10} = 37,$$

subject to the restriction

$$u_1, \dots, u_5 \in \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad u_6, \dots, u_9 \in \{2, 3, 4, 5, 6, 7, 8\} \quad \text{and} \quad u_{10} \in \{5, 10, 15, \dots\},$$

is given by the coefficient of  $X^{37}$  in the formal power series

$$\begin{aligned} f(X) &= (X^1 + \dots + X^6)^5 (X^2 + \dots + X^8)^4 (X^5 + X^{10} + X^{15} + \dots) \\ &= \left( \frac{X - X^7}{1 - X} \right)^5 \left( \frac{X^2 - X^9}{1 - X} \right)^4 \left( \frac{X^5}{1 - X^5} \right) = X^{18} (1 - X^6)^5 (1 - X^7)^4 (1 - X)^{-9} (1 - X^5)^{-1}; \end{aligned}$$

or the coefficient of  $X^{19}$  in the formal power series

$$g(X) = (1 - X^6)^5(1 - X^7)^4(1 - X)^{-9}(1 - X^5)^{-1} = h(X)(1 - X)^{-9}(1 - X^5)^{-1},$$

where

$$\begin{aligned} h(X) &= (1 - X^6)^5(1 - X^7)^4 \\ &= (1 - 5X^6 + 10X^{12} - 10X^{18} + 5X^{24} - X^{30})(1 - 4X^7 + 6X^{14} - 4X^{21} + X^{28}) \\ &= 1 - 5X^6 - 4X^7 + 10X^{12} + 20X^{13} + 6X^{14} - 10X^{18} - 40X^{19} \\ &\quad + \text{terms with higher powers in } X. \end{aligned}$$

It follows that the coefficient of  $X^{19}$  in  $g(X)$  is given by

$$\begin{aligned} &1 \times \text{coefficient of } X^{19} \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} - 5 \times \text{coefficient of } X^{13} \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} \\ &- 4 \times \text{coefficient of } X^{12} \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} + 10 \times \text{coefficient of } X^7 \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} \\ &+ 20 \times \text{coefficient of } X^6 \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} + 6 \times \text{coefficient of } X^5 \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} \\ &- 10 \times \text{coefficient of } X^1 \text{ in } (1 - X)^{-9}(1 - X^5)^{-1} - 40 \times \text{coefficient of } X^0 \text{ in } (1 - X)^{-9}(1 - X^5)^{-1}. \end{aligned}$$

This is equal to

$$\begin{aligned} &\left( (-1)^{19} \binom{-9}{19} (-1)^0 \binom{-1}{0} + (-1)^{14} \binom{-9}{14} (-1)^1 \binom{-1}{1} + (-1)^9 \binom{-9}{9} (-1)^2 \binom{-1}{2} \right. \\ &\quad \left. + (-1)^4 \binom{-9}{4} (-1)^3 \binom{-1}{3} \right) \\ &- 5 \left( (-1)^{13} \binom{-9}{13} (-1)^0 \binom{-1}{0} + (-1)^8 \binom{-9}{8} (-1)^1 \binom{-1}{1} + (-1)^3 \binom{-9}{3} (-1)^2 \binom{-1}{2} \right) \\ &- 4 \left( (-1)^{12} \binom{-9}{12} (-1)^0 \binom{-1}{0} + (-1)^7 \binom{-9}{7} (-1)^1 \binom{-1}{1} + (-1)^2 \binom{-9}{2} (-1)^2 \binom{-1}{2} \right) \\ &+ 10 \left( (-1)^7 \binom{-9}{7} (-1)^0 \binom{-1}{0} + (-1)^2 \binom{-9}{2} (-1)^1 \binom{-1}{1} \right) \\ &+ 20 \left( (-1)^6 \binom{-9}{6} (-1)^0 \binom{-1}{0} + (-1)^1 \binom{-9}{1} (-1)^1 \binom{-1}{1} \right) \\ &+ 6 \left( (-1)^5 \binom{-9}{5} (-1)^0 \binom{-1}{0} + (-1)^0 \binom{-9}{0} (-1)^1 \binom{-1}{1} \right) \\ &- 10 \left( (-1)^1 \binom{-9}{1} (-1)^0 \binom{-1}{0} \right) \\ &- 40 (-1)^0 \binom{-9}{0} (-1)^0 \binom{-1}{0} \\ &= \left( \binom{27}{19} + \binom{22}{14} + \binom{17}{9} + \binom{12}{4} \right) - 5 \left( \binom{21}{13} + \binom{16}{8} + \binom{11}{3} \right) - 4 \left( \binom{20}{12} + \binom{15}{7} + \binom{10}{2} \right) \\ &\quad + 10 \left( \binom{15}{7} + \binom{10}{2} \right) + 20 \left( \binom{14}{6} + \binom{9}{1} \right) + 6 \left( \binom{13}{5} + \binom{8}{0} \right) - 10 \binom{9}{1} - 40 \binom{8}{0}. \end{aligned}$$

Let us reconsider this same question by using the Inclusion-exclusion principle. In view of Proposition 15B, the problem is similar to the problem of finding the number of solutions of the equation

$$v_1 + \dots + v_{10} = 19,$$

subject to the restrictions  $v_1, \dots, v_5 \in \{0, 1, 2, 3, 4, 5\}$ ,  $v_6, \dots, v_9 \in \{0, 1, 2, 3, 4, 5, 6\}$  and  $v_{10} \in \{0, 5, 10, 15, \dots\}$ . Clearly we must have four cases: (I)  $v_{10} = 0$ ; (II)  $v_{10} = 5$ ; (III)  $v_{10} = 10$ ; and (IV)  $v_{10} = 15$ . The number of case (I) solutions is equal to the number of solutions of the equation

$$v_1 + \dots + v_9 = 19,$$

subject to the restriction

$$(38) \quad v_1, \dots, v_5 \in \{0, 1, 2, 3, 4, 5\} \quad \text{and} \quad v_6, \dots, v_9 \in \{0, 1, 2, 3, 4, 5, 6\}.$$

Using the Inclusion-exclusion principle, we can show that the number of solutions is given by

$$\binom{27}{19} - 5\binom{21}{13} - 4\binom{20}{12} + 10\binom{15}{7} + 20\binom{14}{6} + 6\binom{13}{5} - 10\binom{9}{1} - 40\binom{8}{0}.$$

The number of case (II) solutions is equal to the number of solutions of the equation

$$v_1 + \dots + v_9 = 14,$$

subject to the restriction (38). This can be shown to be

$$\binom{22}{14} - 5\binom{16}{8} - 4\binom{15}{7} + 10\binom{10}{2} + 20\binom{9}{1} + 6\binom{8}{0}.$$

The number of case (III) solutions is equal to the number of solutions of the equation

$$v_1 + \dots + v_9 = 9,$$

subject to the restriction (38). This can be shown to be

$$\binom{17}{9} - 5\binom{11}{3} - 4\binom{10}{2}.$$

Finally, the number of case (IV) solutions is equal to the number of solutions of the equation

$$v_1 + \dots + v_9 = 4,$$

subject to the restriction (38). This can be shown to be

$$\binom{12}{4}.$$

#### PROBLEMS FOR CHAPTER 15

- Consider the linear equation  $u_1 + \dots + u_6 = 29$ . Use the arguments in Sections 15.2–15.4 to solve the following problems:
  - How many solutions satisfy  $u_1, \dots, u_6 \in \{0, 1, 2, 3, \dots\}$ ?
  - How many solutions satisfy  $u_1, \dots, u_6 \in \{0, 1, 2, 3, \dots, 7\}$ ?
  - How many solutions satisfy  $u_1, \dots, u_6 \in \{2, 3, \dots, 7\}$ ?
  - How many solutions satisfy  $u_1, u_2, u_3 \in \{1, 2, 3, \dots, 7\}$  and  $u_4, u_5, u_6 \in \{2, 3, \dots, 6\}$ ?
- Consider the linear equation  $u_1 + \dots + u_7 = 34$ . Use the arguments in Sections 15.2–15.4 to solve the following problems:
  - How many solutions satisfy  $u_1, \dots, u_7 \in \{0, 1, 2, 3, \dots\}$ ?
  - How many solutions satisfy  $u_1, \dots, u_7 \in \{0, 1, 2, 3, \dots, 7\}$ ?
  - How many solutions satisfy  $u_1, \dots, u_7 \in \{2, 3, 4, \dots, 7\}$ ?
  - How many solutions satisfy  $u_1, \dots, u_4 \in \{1, 2, 3, \dots, 8\}$  and  $u_5, \dots, u_7 \in \{3, 4, 5, \dots, 7\}$ ?
- How many natural numbers  $x \leq 99999999$  are such that the sum of the digits in  $x$  is equal to 35? How many of these numbers are 8-digit numbers?

4. Rework Questions 1 and 2 using generating functions.
5. Use generating functions to find the number of solutions of the linear equation  $u_1 + \dots + u_5 = 25$  subject to the restrictions that  $u_1, \dots, u_4 \in \{1, 2, 3, \dots, 8\}$  and  $u_5 \in \{2, 3, 13\}$ .
6. Consider again the linear equation  $u_1 + \dots + u_7 = 34$ . How many solutions of this equation satisfy  $u_1, \dots, u_6 \in \{1, 2, 3, \dots, 8\}$  and  $u_7 \in \{2, 3, 5, 7\}$ ?