

DISCRETE MATHEMATICS

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Chapter 16

RECURRENCE RELATIONS

16.1. Introduction

Any equation involving several terms of a sequence is called a recurrence relation. We shall think of the integer n as the independent variable, and restrict our attention to real sequences, so that the sequence a_n is considered as a function of the type

$$f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R} : n \mapsto a_n.$$

A recurrence relation is then an equation of the type

$$F(n, a_n, a_{n+1}, \dots, a_{n+k}) = 0,$$

where $k \in \mathbb{N}$ is fixed.

EXAMPLE 16.1.1. $a_{n+1} = 5a_n$ is a recurrence relation of order 1.

EXAMPLE 16.1.2. $a_{n+1}^4 + a_n^5 = n$ is a recurrence relation of order 1.

EXAMPLE 16.1.3. $a_{n+3} + 5a_{n+2} + 4a_{n+1} + a_n = \cos n$ is a recurrence relation of order 3.

EXAMPLE 16.1.4. $a_{n+2} + 5(a_{n+1}^2 + a_n)^{1/3} = 0$ is a recurrence relation of order 2.

We now define the order of a recurrence relation.

DEFINITION. The order of a recurrence relation is the difference between the greatest and lowest subscripts of the terms of the sequence in the equation.

DEFINITION. A recurrence relation of order k is said to be linear if it is linear in $a_n, a_{n+1}, \dots, a_{n+k}$. Otherwise, the recurrence relation is said to be non-linear.

EXAMPLE 16.1.5. The recurrence relations in Examples 16.1.1 and 16.1.3 are linear, while those in Examples 16.1.2 and 16.1.4 are non-linear.

EXAMPLE 16.1.6. $a_{n+1}a_{n+2} = 5a_n$ is a non-linear recurrence relation of order 2.

REMARK. The recurrence relation $a_{n+3} + 5a_{n+2} + 4a_{n+1} + a_n = \cos n$ can also be written in the form $a_{n+2} + 5a_{n+1} + 4a_n + a_{n-1} = \cos(n-1)$. There is no reason why the term of the sequence in the equation with the lowest subscript should always have subscript n .

For the sake of uniformity and convenience, we shall in this chapter always follow the convention that the term of the sequence in the equation with the lowest subscript has subscript n .

16.2. How Recurrence Relations Arise

We shall first of all consider a few examples. Do not worry about the details.

EXAMPLE 16.2.1. Consider the equation $a_n = A(n!)$, where A is a constant. Replacing n by $(n+1)$ in the equation, we obtain $a_{n+1} = A((n+1)!)$. Combining the two equations and eliminating A , we obtain the first-order recurrence relation $a_{n+1} = (n+1)a_n$.

EXAMPLE 16.2.2. Consider the equation

$$(1) \quad a_n = (A + Bn)3^n,$$

where A and B are constants. Replacing n by $(n+1)$ and $(n+2)$ in the equation, we obtain respectively

$$(2) \quad a_{n+1} = (A + B(n+1))3^{n+1} = (3A + 3B)3^n + 3Bn3^n$$

and

$$(3) \quad a_{n+2} = (A + B(n+2))3^{n+2} = (9A + 18B)3^n + 9Bn3^n.$$

Combining (1)–(3) and eliminating A and B , we obtain the second-order recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0.$$

EXAMPLE 16.2.3. Consider the equation

$$(4) \quad a_n = A(-1)^n + B(-2)^n + C3^n,$$

where A , B and C are constants. Replacing n by $(n+1)$, $(n+2)$ and $(n+3)$ in the equation, we obtain respectively

$$(5) \quad a_{n+1} = A(-1)^{n+1} + B(-2)^{n+1} + C3^{n+1} = -A(-1)^n - 2B(-2)^n + 3C3^n,$$

$$(6) \quad a_{n+2} = A(-1)^{n+2} + B(-2)^{n+2} + C3^{n+2} = A(-1)^n + 4B(-2)^n + 9C3^n$$

and

$$(7) \quad a_{n+3} = A(-1)^{n+3} + B(-2)^{n+3} + C3^{n+3} = -A(-1)^n - 8B(-2)^n + 27C3^n.$$

Combining (4)–(7) and eliminating A , B and C , we obtain the third-order recurrence relation

$$a_{n+3} - 7a_{n+1} - 6a_n = 0.$$

Note that in these three examples, the expression of a_n as a function of n contains respectively one, two and three constants. By writing down one, two and three extra expressions respectively, using subsequent terms of the sequence, we are in a position to eliminate these constants.

In general, the expression of a_n as a function of n may contain k arbitrary constants. By writing down k further equations, using subsequent terms of the sequence, we expect to be able to eliminate these constants. After eliminating these constants, we expect to end up with a recurrence relation of order k .

If we reverse the argument, it is reasonable to define the general solution of a recurrence relation of order k as that solution containing k arbitrary constants. This is, however, not very satisfactory. Instead, the following is true: Any solution of a recurrence relation of order k containing fewer than k arbitrary constants cannot be the general solution.

In many situations, the solution of a recurrence relation has to satisfy certain specified conditions. These are called initial conditions, and determine the values of the arbitrary constants in the solution.

EXAMPLE 16.2.4. The recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

has general solution $a_n = (A + Bn)3^n$, where A and B are arbitrary constants. Suppose that we have the initial conditions $a_0 = 1$ and $a_1 = 15$. Then we must have $a_n = (1 + 4n)3^n$.

16.3. Linear Recurrence Relations

Non-linear recurrence relations are usually very difficult, with standard techniques only for very few cases. We shall therefore concentrate on linear recurrence relations.

The general linear recurrence relation of order k is the equation

$$(8) \quad s_0(n)a_{n+k} + s_1(n)a_{n+k-1} + \dots + s_k(n)a_n = f(n),$$

where $s_0(n), s_1(n), \dots, s_k(n)$ and $f(n)$ are given functions. Here we are primarily concerned with (8) only when the coefficients $s_0(n), s_1(n), \dots, s_k(n)$ are constants and hence independent of n . We therefore study equations of the type

$$(9) \quad s_0a_{n+k} + s_1a_{n+k-1} + \dots + s_ka_n = f(n),$$

where s_0, s_1, \dots, s_k are constants, and where $f(n)$ is a given function.

16.4. The Homogeneous Case

If the function $f(n)$ on the right-hand side of (9) is identically zero, then we say that the recurrence relation (9) is homogeneous. If the function $f(n)$ on the right-hand side of (9) is not identically zero, then we say that the recurrence relation

$$(10) \quad s_0a_{n+k} + s_1a_{n+k-1} + \dots + s_ka_n = 0$$

is the reduced recurrence relation of (9).

In this section, we study the problem of finding the general solution of a homogeneous recurrence relation of the type (10).

Suppose that $a_n^{(1)}, \dots, a_n^{(k)}$ are k independent solutions of the recurrence relation (10), so that no linear combination of them with constant coefficients is identically zero, and

$$s_0 a_{n+k}^{(1)} + s_1 a_{n+k-1}^{(1)} + \dots + s_k a_n^{(1)} = 0, \quad \dots, \quad s_0 a_{n+k}^{(k)} + s_1 a_{n+k-1}^{(k)} + \dots + s_k a_n^{(k)} = 0.$$

We consider the linear combination

$$(11) \quad a_n = c_1 a_n^{(1)} + \dots + c_k a_n^{(k)},$$

where c_1, \dots, c_k are arbitrary constants. Then a_n is clearly also a solution of (10), for

$$\begin{aligned} & s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n \\ &= s_0 (c_1 a_{n+k}^{(1)} + \dots + c_k a_{n+k}^{(k)}) + s_1 (c_1 a_{n+k-1}^{(1)} + \dots + c_k a_{n+k-1}^{(k)}) + \dots + s_k (c_1 a_n^{(1)} + \dots + c_k a_n^{(k)}) \\ &= c_1 (s_0 a_{n+k}^{(1)} + s_1 a_{n+k-1}^{(1)} + \dots + s_k a_n^{(1)}) + \dots + c_k (s_0 a_{n+k}^{(k)} + s_1 a_{n+k-1}^{(k)} + \dots + s_k a_n^{(k)}) \\ &= 0. \end{aligned}$$

Since (11) contains k constants, it is reasonable to take this as the general solution of (10). It remains to find k independent solutions $a_n^{(1)}, \dots, a_n^{(k)}$.

Consider first of all the case $k = 2$. We are therefore interested in the homogeneous recurrence relation

$$(12) \quad s_0 a_{n+2} + s_1 a_{n+1} + s_2 a_n = 0,$$

where s_0, s_1, s_2 are constants, with $s_0 \neq 0$ and $s_2 \neq 0$. Let us try a solution of the form

$$(13) \quad a_n = \lambda^n,$$

where $\lambda \neq 0$. Then clearly $a_{n+1} = \lambda^{n+1}$ and $a_{n+2} = \lambda^{n+2}$, so that

$$(s_0 \lambda^2 + s_1 \lambda + s_2) \lambda^n = 0.$$

Since $\lambda \neq 0$, we must have

$$(14) \quad s_0 \lambda^2 + s_1 \lambda + s_2 = 0.$$

This is called the characteristic polynomial of the recurrence relation (12).

It follows that (13) is a solution of the recurrence relation (12) whenever λ satisfies the characteristic polynomial (14). Suppose that λ_1 and λ_2 are the two roots of (14). Then

$$a_n^{(1)} = \lambda_1^n \quad \text{and} \quad a_n^{(2)} = \lambda_2^n$$

are both solutions of the recurrence relation (12). It follows that the general solution of the recurrence relation (12) is

$$(15) \quad a_n = c_1 \lambda_1^n + c_2 \lambda_2^n.$$

EXAMPLE 16.4.1. The recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 0$$

has characteristic polynomial $\lambda^2 + 4\lambda + 3 = 0$, with roots $\lambda_1 = -3$ and $\lambda_2 = -1$. It follows that the general solution of the recurrence relation is given by

$$a_n = c_1 (-3)^n + c_2 (-1)^n.$$

EXAMPLE 16.4.2. The recurrence relation

$$a_{n+2} + 4a_n = 0$$

has characteristic polynomial $\lambda^2 + 4 = 0$, with roots $\lambda_1 = 2i$ and $\lambda_2 = -2i$. It follows that the general solution of the recurrence relation is given by

$$\begin{aligned} a_n &= b_1(2i)^n + b_2(-2i)^n = 2^n(b_1i^n + b_2(-i)^n) \\ &= 2^n \left(b_1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^n + b_2 \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^n \right) \\ &= 2^n \left(b_1 \left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} \right) + b_2 \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right) \right) \\ &= 2^n \left((b_1 + b_2) \cos \frac{n\pi}{2} + i(b_1 - b_2) \sin \frac{n\pi}{2} \right) \\ &= 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \right). \end{aligned}$$

EXAMPLE 16.4.3. The recurrence relation

$$a_{n+2} + 4a_{n+1} + 16a_n = 0$$

has characteristic polynomial $\lambda^2 + 4\lambda + 16 = 0$, with roots $\lambda_1 = -2 + 2\sqrt{3}i$ and $\lambda_2 = -2 - 2\sqrt{3}i$. It follows that the general solution of the recurrence relation is given by

$$\begin{aligned} a_n &= b_1(-2 + 2\sqrt{3}i)^n + b_2(-2 - 2\sqrt{3}i)^n = 4^n \left(b_1 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^n + b_2 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)^n \right) \\ &= 4^n \left(b_1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^n + b_2 \left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right)^n \right) \\ &= 4^n \left(b_1 \left(\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right) + b_2 \left(\cos \frac{2n\pi}{3} - i \sin \frac{2n\pi}{3} \right) \right) \\ &= 4^n \left((b_1 + b_2) \cos \frac{2n\pi}{3} + i(b_1 - b_2) \sin \frac{2n\pi}{3} \right) \\ &= 4^n \left(c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} \right). \end{aligned}$$

The method works well provided that $\lambda_1 \neq \lambda_2$. However, if $\lambda_1 = \lambda_2$, then (15) does not qualify as the general solution of the recurrence relation (12), as it contains only one arbitrary constant. We therefore try for a solution of the form

$$(16) \quad a_n = u_n \lambda^n,$$

where u_n is a function of n , and where λ is the repeated root of the characteristic polynomial (14). Then

$$(17) \quad a_{n+1} = u_{n+1} \lambda^{n+1} \quad \text{and} \quad a_{n+2} = u_{n+2} \lambda^{n+2}.$$

Substituting (16) and (17) into (12), we obtain

$$s_0 u_{n+2} \lambda^{n+2} + s_1 u_{n+1} \lambda^{n+1} + s_2 u_n \lambda^n = 0.$$

Note that the left-hand side is equal to

$$s_0(u_{n+2} - u_n) \lambda^{n+2} + s_1(u_{n+1} - u_n) \lambda^{n+1} + u_n(s_0 \lambda^2 + s_1 \lambda + s_2) \lambda^n = s_0(u_{n+2} - u_n) \lambda^{n+2} + s_1(u_{n+1} - u_n) \lambda^{n+1}.$$

It follows that

$$s_0\lambda(u_{n+2} - u_n) + s_1(u_{n+1} - u_n) = 0.$$

Note now that since $s_0\lambda^2 + s_1\lambda + s_2 = 0$ and that λ is a repeated root, we must have $2\lambda = -s_1/s_0$. It follows that we must have $(u_{n+2} - u_n) - 2(u_{n+1} - u_n) = 0$, so that

$$u_{n+2} - 2u_{n+1} + u_n = 0.$$

This implies that the sequence u_n is an arithmetic progression, so that $u_n = c_1 + c_2n$, where c_1 and c_2 are constants. It follows that the general solution of the recurrence relation (12) in this case is given by

$$a_n = (c_1 + c_2n)\lambda^n,$$

where λ is the repeated root of the characteristic polynomial (14).

EXAMPLE 16.4.4. The recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

has characteristic polynomial $\lambda^2 - 6\lambda + 9 = 0$, with repeated roots $\lambda = 3$. It follows that the general solution of the recurrence relation is given by

$$a_n = (c_1 + c_2n)3^n.$$

We now consider the general case. We are therefore interested in the homogeneous recurrence relation

$$(18) \quad s_0a_{n+k} + s_1a_{n+k-1} + \dots + s_ka_n = 0,$$

where s_0, s_1, \dots, s_k are constants, with $s_0 \neq 0$. If we try a solution of the form $a_n = \lambda^n$ as before, where $\lambda \neq 0$, then it can easily be shown that we must have

$$(19) \quad s_0\lambda^k + s_1\lambda^{k-1} + \dots + s_k = 0.$$

This is called the characteristic polynomial of the recurrence relation (18).

We shall state the following theorem without proof.

PROPOSITION 16A. *Suppose that the characteristic polynomial (19) of the homogeneous recurrence relation (18) has distinct roots $\lambda_1, \dots, \lambda_s$, with multiplicities m_1, \dots, m_s respectively (where, of course, $k = m_1 + \dots + m_s$). Then the general solution of the recurrence relation (18) is given by*

$$a_n = \sum_{j=1}^s (b_{j,1} + b_{j,2}n + \dots + b_{j,m_j}n^{m_j-1})\lambda_j^n,$$

where, for every $j = 1, \dots, s$, the coefficients $b_{j,1}, \dots, b_{j,m_j}$ are constants.

EXAMPLE 16.4.5. The recurrence relation

$$a_{n+5} + 7a_{n+4} + 19a_{n+3} + 25a_{n+2} + 16a_{n+1} + 4a_n = 0$$

has characteristic polynomial $\lambda^5 + 7\lambda^4 + 19\lambda^3 + 25\lambda^2 + 16\lambda + 4 = (\lambda + 1)^3(\lambda + 2)^2 = 0$ with roots $\lambda_1 = -1$ and $\lambda_2 = -2$ with multiplicities $m_1 = 3$ and $m_2 = 2$ respectively. It follows that the general solution of the recurrence relation is given by

$$a_n = (c_1 + c_2n + c_3n^2)(-1)^n + (c_4 + c_5n)(-2)^n.$$

16.5. The Non-Homogeneous Case

We study equations of the type

$$(20) \quad s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n = f(n),$$

where s_0, s_1, \dots, s_k are constants, and where $f(n)$ is a given function.

Suppose that $a_n^{(c)}$ is the general solution of the reduced recurrence relation

$$(21) \quad s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n = 0,$$

so that the expression of $a_n^{(c)}$ involves k arbitrary constants. Suppose further that $a_n^{(p)}$ is any solution of the non-homogeneous recurrence relation (20). Then

$$s_0 a_{n+k}^{(c)} + s_1 a_{n+k-1}^{(c)} + \dots + s_k a_n^{(c)} = 0 \quad \text{and} \quad s_0 a_{n+k}^{(p)} + s_1 a_{n+k-1}^{(p)} + \dots + s_k a_n^{(p)} = f(n).$$

Let

$$(22) \quad a_n = a_n^{(c)} + a_n^{(p)}.$$

Then

$$\begin{aligned} & s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n \\ &= s_0 (a_{n+k}^{(c)} + a_{n+k}^{(p)}) + s_1 (a_{n+k-1}^{(c)} + a_{n+k-1}^{(p)}) + \dots + s_k (a_n^{(c)} + a_n^{(p)}) \\ &= (s_0 a_{n+k}^{(c)} + s_1 a_{n+k-1}^{(c)} + \dots + s_k a_n^{(c)}) + (s_0 a_{n+k}^{(p)} + s_1 a_{n+k-1}^{(p)} + \dots + s_k a_n^{(p)}) \\ &= 0 + f(n) = f(n). \end{aligned}$$

It is therefore reasonable to say that (22) is the general solution of the non-homogeneous recurrence relation (20).

The term $a_n^{(c)}$ is usually known as the complementary function of the recurrence relation (20), while the term $a_n^{(p)}$ is usually known as a particular solution of the recurrence relation (20). Note that $a_n^{(p)}$ is in general not unique.

To solve the recurrence relation (20), it remains to find a particular solution $a_n^{(p)}$.

16.6. The Method of Undetermined Coefficients

In this section, we are concerned with the question of finding particular solutions of recurrence relations of the type

$$(23) \quad s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n = f(n),$$

where s_0, s_1, \dots, s_k are constants, and where $f(n)$ is a given function.

The method of undetermined coefficients is based on assuming a trial form for the particular solution $a_n^{(p)}$ of (23) which depends on the form of the function $f(n)$ and which contains a number of arbitrary constants. This trial function is then substituted into the recurrence relation (23) and the constants are chosen to make this a solution.

The basic trial forms are given in the table below (c denotes a constant in the expression of $f(n)$ and A (with or without subscripts) denotes a constant to be determined):

$f(n)$	trial $a_n^{(p)}$	$f(n)$	trial $a_n^{(p)}$
c	A	$c \sin \alpha n$	$A_1 \cos \alpha n + A_2 \sin \alpha n$
cn	$A_0 + A_1 n$	$c \cos \alpha n$	$A_1 \cos \alpha n + A_2 \sin \alpha n$
cn^2	$A_0 + A_1 n + A_2 n^2$	$cr^n \sin \alpha n$	$A_1 r^n \cos \alpha n + A_2 r^n \sin \alpha n$
cn^m ($m \in \mathbb{N}$)	$A_0 + A_1 n + \dots + A_m n^m$	$cr^n \cos \alpha n$	$A_1 r^n \cos \alpha n + A_2 r^n \sin \alpha n$
cr^n ($r \in \mathbb{R}$)	Ar^n	$cn^m r^n$	$r^n (A_0 + A_1 n + \dots + A_m n^m)$

EXAMPLE 16.6.1. Consider the recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n.$$

It has been shown in Example 16.4.1 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = c_1(-3)^n + c_2(-1)^n.$$

For a particular solution, we try

$$a_n^{(p)} = A(-2)^n.$$

Then

$$a_{n+1}^{(p)} = A(-2)^{n+1} = -2A(-2)^n$$

and

$$a_{n+2}^{(p)} = A(-2)^{n+2} = 4A(-2)^n.$$

It follows that

$$a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 3a_n^{(p)} = (4A - 8A + 3A)(-2)^n = -A(-2)^n = 5(-2)^n$$

if $A = -5$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = c_1(-3)^n + c_2(-1)^n - 5(-2)^n.$$

EXAMPLE 16.6.2. Consider the recurrence relation

$$a_{n+2} + 4a_n = 6 \cos \frac{n\pi}{2} + 3 \sin \frac{n\pi}{2}.$$

It has been shown in Example 16.4.2 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \right).$$

For a particular solution, we try

$$a_n^{(p)} = A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2}.$$

Then

$$\begin{aligned} a_{n+1}^{(p)} &= A_1 \cos \frac{(n+1)\pi}{2} + A_2 \sin \frac{(n+1)\pi}{2} \\ &= A_1 \left(\cos \frac{n\pi}{2} \cos \frac{\pi}{2} - \sin \frac{n\pi}{2} \sin \frac{\pi}{2} \right) + A_2 \left(\sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos \frac{n\pi}{2} \sin \frac{\pi}{2} \right) \\ &= A_2 \cos \frac{n\pi}{2} - A_1 \sin \frac{n\pi}{2} \end{aligned}$$

and

$$\begin{aligned} a_{n+2}^{(p)} &= A_1 \cos \frac{(n+2)\pi}{2} + A_2 \sin \frac{(n+2)\pi}{2} \\ &= A_1 \left(\cos \frac{n\pi}{2} \cos \pi - \sin \frac{n\pi}{2} \sin \pi \right) + A_2 \left(\sin \frac{n\pi}{2} \cos \pi + \cos \frac{n\pi}{2} \sin \pi \right) \\ &= -A_1 \cos \frac{n\pi}{2} - A_2 \sin \frac{n\pi}{2}. \end{aligned}$$

It follows that

$$a_{n+2}^{(p)} + 4a_n^{(p)} = 3A_1 \cos \frac{n\pi}{2} + 3A_2 \sin \frac{n\pi}{2} = 6 \cos \frac{n\pi}{2} + 3 \sin \frac{n\pi}{2}$$

if $A_1 = 2$ and $A_2 = 1$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \right) + 2 \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}.$$

EXAMPLE 16.6.3. Consider the recurrence relation

$$a_{n+2} + 4a_{n+1} + 16a_n = 4^{n+2} \cos \frac{n\pi}{2} - 4^{n+3} \sin \frac{n\pi}{2}.$$

It has been shown in Example 16.4.3 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = 4^n \left(c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} \right).$$

For a particular solution, we try

$$a_n^{(p)} = 4^n \left(A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2} \right).$$

Then

$$a_{n+1}^{(p)} = 4^{n+1} \left(A_1 \cos \frac{(n+1)\pi}{2} + A_2 \sin \frac{(n+1)\pi}{2} \right) = 4^n \left(4A_2 \cos \frac{n\pi}{2} - 4A_1 \sin \frac{n\pi}{2} \right)$$

and

$$a_{n+2}^{(p)} = 4^{n+2} \left(A_1 \cos \frac{(n+2)\pi}{2} + A_2 \sin \frac{(n+2)\pi}{2} \right) = 4^n \left(-16A_1 \cos \frac{n\pi}{2} - 16A_2 \sin \frac{n\pi}{2} \right).$$

It follows that

$$a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 16a_n^{(p)} = 16A_2 4^n \cos \frac{n\pi}{2} - 16A_1 4^n \sin \frac{n\pi}{2} = 4^{n+2} \cos \frac{n\pi}{2} - 4^{n+3} \sin \frac{n\pi}{2}$$

if $A_1 = 4$ and $A_2 = 1$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = 4^n \left(c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} + 4 \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right).$$

16.7. Lifting the Trial Functions

What we have discussed so far in Section 16.6 may not work. We need to find a remedy.

EXAMPLE 16.7.1. Consider the recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 12(-3)^n.$$

It has been shown in Example 16.4.1 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = c_1(-3)^n + c_2(-1)^n.$$

For a particular solution, let us try

$$a_n^{(p)} = A(-3)^n.$$

Then

$$a_{n+1}^{(p)} = A(-3)^{n+1} = -3A(-3)^n$$

and

$$a_{n+2}^{(p)} = A(-3)^{n+2} = 9A(-3)^n.$$

But

$$a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 3a_n^{(p)} = (9A - 12A + 3A)(-3)^n = 0 \neq 12(-3)^n$$

for any A . In fact, this is no coincidence. Note that if we take $c_1 = A$ and $c_2 = 0$, then the complementary function $a_n^{(c)}$ becomes our trial function! No wonder the method does not work. Now try instead

$$a_n^{(p)} = An(-3)^n.$$

Then

$$a_{n+1}^{(p)} = A(n+1)(-3)^{n+1} = -3An(-3)^n - 3A(-3)^n$$

and

$$a_{n+2}^{(p)} = A(n+2)(-3)^{n+2} = 9An(-3)^n + 18A(-3)^n.$$

It follows that

$$a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 3a_n^{(p)} = (9A - 12A + 3A)n(-3)^n + (18A - 12A)(-3)^n = 6A(-3)^n = 12(-3)^n$$

if $A = 2$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = c_1(-3)^n + c_2(-1)^n + 2n(-3)^n.$$

EXAMPLE 16.7.2. Consider the recurrence relation

$$a_{n+2} + 4a_n = 2^n \cos \frac{n\pi}{2}.$$

It has been shown in Example 16.4.2 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \right).$$

For a particular solution, it is no use trying

$$a_n^{(p)} = 2^n \left(A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2} \right).$$

It is guaranteed not to work. Now try instead

$$a_n^{(p)} = n2^n \left(A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2} \right).$$

Then

$$\begin{aligned} a_{n+1}^{(p)} &= (n+1)2^{n+1} \left(A_1 \cos \frac{(n+1)\pi}{2} + A_2 \sin \frac{(n+1)\pi}{2} \right) \\ &= (n+1)2^{n+1} \left(A_2 \cos \frac{n\pi}{2} - A_1 \sin \frac{n\pi}{2} \right) \end{aligned}$$

and

$$\begin{aligned} a_{n+2}^{(p)} &= (n+2)2^{n+2} \left(A_1 \cos \frac{(n+2)\pi}{2} + A_2 \sin \frac{(n+2)\pi}{2} \right) \\ &= (n+2)2^{n+2} \left(-A_1 \cos \frac{n\pi}{2} - A_2 \sin \frac{n\pi}{2} \right). \end{aligned}$$

It follows that

$$\begin{aligned} a_{n+2}^{(p)} + 4a_n^{(p)} &= (-(n+2)2^{n+2} + 4n2^n)A_1 \cos \frac{n\pi}{2} + (-(n+2)2^{n+2} + 4n2^n)A_2 \sin \frac{n\pi}{2} \\ &= -2^{n+3}A_1 \cos \frac{n\pi}{2} - 2^{n+3}A_2 \sin \frac{n\pi}{2} = 2^n \cos \frac{n\pi}{2} \end{aligned}$$

if $A_1 = -1/8$ and $A_2 = 0$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} - \frac{n}{8} \cos \frac{n\pi}{2} \right).$$

EXAMPLE 16.7.3. Consider the recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 3^n.$$

It has been shown in Example 16.4.4 that the reduced recurrence relation has complementary function

$$a_n^{(c)} = (c_1 + c_2 n)3^n.$$

For a particular solution, it is no use trying

$$a_n^{(p)} = A3^n \quad \text{or} \quad a_n^{(p)} = An3^n.$$

Both are guaranteed not to work. Now try instead

$$a_n^{(p)} = An^2 3^n.$$

Then

$$a_{n+1}^{(p)} = A(n+1)^2 3^{n+1} = A(3n^2 + 6n + 3)3^n$$

and

$$a_{n+2}^{(p)} = A(n+2)^2 3^{n+2} = A(9n^2 + 36n + 36)3^n.$$

It follows that

$$a_{n+2}^{(p)} - 6a_{n+1}^{(p)} + 9a_n^{(p)} = 18A3^n = 3^n$$

if $A = 1/18$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = \left(c_1 + c_2n + \frac{n^2}{18} \right) 3^n.$$

In general, all we need to do when the usual trial function forms part of the complementary function is to “lift our usual trial function over the complementary function” by multiplying the usual trial function by a power of n . This power should be as small as possible.

16.8. Initial Conditions

We shall illustrate our method by a fresh example.

EXAMPLE 16.8.1. Consider the recurrence relation

$$a_{n+3} + 5a_{n+2} + 8a_{n+1} + 4a_n = 2(-1)^n + (-2)^{n+3},$$

with initial conditions $a_0 = 4$, $a_1 = -11$ and $a_2 = 41$. Consider first of all the reduced recurrence relation

$$a_{n+3} + 5a_{n+2} + 8a_{n+1} + 4a_n = 0.$$

This has characteristic polynomial

$$\lambda^3 + 5\lambda^2 + 8\lambda + 4 = (\lambda + 1)(\lambda + 2)^2 = 0,$$

with roots $\lambda_1 = -1$ and $\lambda_2 = -2$, with multiplicities $m_1 = 1$ and $m_2 = 2$ respectively. It follows that

$$a_n^{(c)} = c_1(-1)^n + (c_2 + c_3n)(-2)^n.$$

To find a particular solution, we therefore need to try

$$a_n^{(p)} = A_1n(-1)^n + A_2n^2(-2)^n.$$

Note that the usual trial function $A_1(-1)^n + A_2(-2)^n$ has been lifted in view of the observation that it forms part of the complementary function. The part $A_1(-1)^n$ has been lifted once since -1 is a root of multiplicity 1 of the characteristic polynomial. The part $A_2(-2)^n$ has been lifted twice since -2 is a root of multiplicity 2 of the characteristic polynomial. Then we have

$$\begin{aligned} a_{n+1}^{(p)} &= A_1(n+1)(-1)^{n+1} + A_2(n+1)^2(-2)^{n+1} \\ &= A_1(-n-1)(-1)^n + A_2(-2n^2 - 4n - 2)(-2)^n, \\ a_{n+2}^{(p)} &= A_1(n+2)(-1)^{n+2} + A_2(n+2)^2(-2)^{n+2} \\ &= A_1(n+2)(-1)^n + A_2(4n^2 + 16n + 16)(-2)^n \end{aligned}$$

and

$$\begin{aligned} a_{n+3}^{(p)} &= A_1(n+3)(-1)^{n+3} + A_2(n+3)^2(-2)^{n+3} \\ &= A_1(-n-3)(-1)^n + A_2(-8n^2 - 48n - 72)(-2)^n. \end{aligned}$$

It follows that

$$a_{n+3}^{(p)} + 5a_{n+2}^{(p)} + 8a_{n+1}^{(p)} + 4a_n^{(p)} = -A_1(-1)^n - 8A_2(-2)^n = 2(-1)^n + (-2)^{n+3}$$

if $A_1 = -2$ and $A_2 = 1$. Hence

$$(24) \quad a_n = a_n^{(c)} + a_n^{(p)} = c_1(-1)^n + (c_2 + c_3n)(-2)^n - 2n(-1)^n + n^2(-2)^n.$$

For the initial conditions to be satisfied, we substitute $n = 0, 1, 2$ into (24) to get respectively

$$\begin{aligned} a_0 &= c_1 + c_2 = 4, \\ a_1 &= -c_1 - 2(c_2 + c_3) + 2 - 2 = -11, \\ a_2 &= c_1 + 4(c_2 + 2c_3) - 4 + 16 = 41. \end{aligned}$$

It follows that we must have $c_1 = 1$, $c_2 = 3$ and $c_3 = 2$, so that

$$a_n = (1 - 2n)(-1)^n + (3 + 2n + n^2)(-2)^n.$$

To summarize, we take the following steps in order:

- (1) Consider the reduced recurrence relation, and find its general solution by finding the roots of its characteristic polynomial. This solution $a_n^{(c)}$ is called the complementary function. If the original recurrence relation is of order k , then the expression for $a_n^{(c)}$ contains k arbitrary constants c_1, \dots, c_k .
- (2) Find a particular solution $a_n^{(p)}$ of the original recurrence relation by using, for example, the method of undetermined coefficients, bearing in mind that in this method, the usual trial function may have to be lifted above the complementary function.
- (3) Obtain the general solution of the original equation by calculating $a_n = a_n^{(c)} + a_n^{(p)}$.
- (4) If initial conditions are given, substitute them into the expression for a_n and determine the constants c_1, \dots, c_k .

16.9. The Generating Function Method

In this section, we are concerned with using generating functions to solve recurrence relations of the type

$$(25) \quad s_0 a_{n+k} + s_1 a_{n+k-1} + \dots + s_k a_n = f(n),$$

where s_0, s_1, \dots, s_k are constants, $f(n)$ is a given function, and the terms a_0, a_1, \dots, a_{k-1} are given.

Let us write

$$(26) \quad G(X) = \sum_{n=0}^{\infty} a_n X^n.$$

In other words, $G(X)$ denotes the generating function of the unknown sequence a_n whose values we wish to determine. If we can determine $G(X)$, then the sequence a_n is simply the sequence of coefficients of the series expansion for $G(X)$.

Multiplying (25) throughout by X^n , we obtain

$$s_0 a_{n+k} X^n + s_1 a_{n+k-1} X^n + \dots + s_k a_n X^n = f(n) X^n.$$

Summing over $n = 0, 1, 2, \dots$, we obtain

$$s_0 \sum_{n=0}^{\infty} a_{n+k} X^n + s_1 \sum_{n=0}^{\infty} a_{n+k-1} X^n + \dots + s_k \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} f(n) X^n.$$

Multiplying throughout again by X^k , we obtain

$$(27) \quad s_0 X^k \sum_{n=0}^{\infty} a_{n+k} X^n + s_1 X^k \sum_{n=0}^{\infty} a_{n+k-1} X^n + \dots + s_k X^k \sum_{n=0}^{\infty} a_n X^n = X^k \sum_{n=0}^{\infty} f(n) X^n.$$

A typical term on the left-hand side of (27) is of the form

$$\begin{aligned} s_j X^k \sum_{n=0}^{\infty} a_{n+k-j} X^n &= s_j X^j \sum_{n=0}^{\infty} a_{n+k-j} X^{n+k-j} = s_j X^j \sum_{m=k-j}^{\infty} a_m X^m \\ &= s_j X^j (G(X) - (a_0 + a_1 X + \dots + a_{k-j-1} X^{k-j-1})), \end{aligned}$$

where $j = 0, 1, \dots, k$. It follows that (27) can be written in the form

$$(28) \quad \sum_{j=0}^k s_j X^j (G(X) - (a_0 + a_1 X + \dots + a_{k-j-1} X^{k-j-1})) = X^k F(X),$$

where

$$F(X) = \sum_{n=0}^{\infty} f(n) X^n$$

is the generating function of the given sequence $f(n)$. Since a_0, a_1, \dots, a_{k-1} are given, it follows that the only unknown in (28) is $G(X)$. Hence we can solve (28) to obtain an expression for $G(X)$.

EXAMPLE 16.9.1. We shall rework Example 16.8.1. We are interested in solving the recurrence relation

$$(29) \quad a_{n+3} + 5a_{n+2} + 8a_{n+1} + 4a_n = 2(-1)^n + (-2)^{n+3},$$

with initial conditions $a_0 = 4$, $a_1 = -11$ and $a_2 = 41$. Multiplying (29) throughout by X^n , we obtain

$$a_{n+3} X^n + 5a_{n+2} X^n + 8a_{n+1} X^n + 4a_n X^n = (2(-1)^n + (-2)^{n+3}) X^n.$$

Summing over $n = 0, 1, 2, \dots$, we obtain

$$\sum_{n=0}^{\infty} a_{n+3} X^n + 5 \sum_{n=0}^{\infty} a_{n+2} X^n + 8 \sum_{n=0}^{\infty} a_{n+1} X^n + 4 \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} (2(-1)^n + (-2)^{n+3}) X^n.$$

Multiplying throughout again by X^3 , we obtain

$$\begin{aligned} X^3 \sum_{n=0}^{\infty} a_{n+3} X^n + 5X^3 \sum_{n=0}^{\infty} a_{n+2} X^n + 8X^3 \sum_{n=0}^{\infty} a_{n+1} X^n + 4X^3 \sum_{n=0}^{\infty} a_n X^n \\ = X^3 \sum_{n=0}^{\infty} (2(-1)^n + (-2)^{n+3}) X^n. \end{aligned}$$

It follows that

$$(30) \quad (G(X) - (a_0 + a_1 X + a_2 X^2)) + 5X(G(X) - (a_0 + a_1 X)) + 8X^2(G(X) - a_0) + 4X^3 G(X) = X^3 F(X),$$

where

$$F(X) = \sum_{n=0}^{\infty} (2(-1)^n + (-2)^{n+3}) X^n$$

is the generating function of the sequence $2(-1)^n + (-2)^{n+3}$. The generating function of the sequence $2(-1)^n$ is

$$F_1(X) = 2 - 2X + 2X^2 - 2X^3 + \dots = 2(1 - X + X^2 - X^3 + \dots) = \frac{2}{1+X},$$

while the generating function of the sequence $(-2)^{n+3}$ is

$$F_2(X) = -8 + 16X - 32X^2 + 64X^3 - \dots = -8(1 - 2X + 4X^2 - 8X^3 + \dots) = -\frac{8}{1+2X}.$$

It follows from Proposition 14A that

$$(31) \quad F(X) = F_1(X) + F_2(X) = \frac{2}{1+X} - \frac{8}{1+2X}.$$

On the other hand, substituting the initial conditions into (30) and combining with (31), we have

$$\begin{aligned} (G(X) - (4 - 11X + 41X^2)) + 5X(G(X) - (4 - 11X)) + 8X^2(G(X) - 4) + 4X^3G(X) \\ = \frac{2X^3}{1+X} - \frac{8X^3}{1+2X}. \end{aligned}$$

In other words,

$$(1 + 5X + 8X^2 + 4X^3)G(X) = \frac{2X^3}{1+X} - \frac{8X^3}{1+2X} + 4 + 9X + 18X^2.$$

Note that $1 + 5X + 8X^2 + 4X^3 = (1+X)(1+2X)^2$, so that

$$(32) \quad G(X) = \frac{2X^3}{(1+X)^2(1+2X)^2} - \frac{8X^3}{(1+X)(1+2X)^3} + \frac{4 + 9X + 18X^2}{(1+X)(1+2X)^2}.$$

This can be expressed by partial fractions in the form

$$\begin{aligned} G(X) &= \frac{A_1}{1+X} + \frac{A_2}{(1+X)^2} + \frac{A_3}{1+2X} + \frac{A_4}{(1+2X)^2} + \frac{A_5}{(1+2X)^3} \\ &= \frac{A_1(1+X)(1+2X)^3 + A_2(1+2X)^3 + A_3(1+X)^2(1+2X)^2 + A_4(1+X)^2(1+2X) + A_5(1+X)^2}{(1+X)^2(1+2X)^3}. \end{aligned}$$

A little calculation from (32) will give

$$G(X) = \frac{4 + 21X + 53X^2 + 66X^3 + 32X^4}{(1+X)^2(1+2X)^3}.$$

It follows that we must have

$$\begin{aligned} A_1(1+X)(1+2X)^3 + A_2(1+2X)^3 + A_3(1+X)^2(1+2X)^2 + A_4(1+X)^2(1+2X) + A_5(1+X)^2 \\ = 4 + 21X + 53X^2 + 66X^3 + 32X^4. \end{aligned}$$

Equating coefficients for X^0, X^1, X^2, X^3, X^4 in the above, we obtain respectively

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 + A_5 &= 4, \\ 7A_1 + 6A_2 + 6A_3 + 4A_4 + 2A_5 &= 21, \\ 18A_1 + 12A_2 + 13A_3 + 5A_4 + A_5 &= 53, \\ 20A_1 + 8A_2 + 12A_3 + 2A_4 &= 66, \\ 8A_1 + 4A_3 &= 32. \end{aligned}$$

This system has unique solution $A_1 = 3$, $A_2 = -2$, $A_3 = 2$, $A_4 = -1$ and $A_5 = 2$, so that

$$G(X) = \frac{3}{1+X} - \frac{2}{(1+X)^2} + \frac{2}{1+2X} - \frac{1}{(1+2X)^2} + \frac{2}{(1+2X)^3}.$$

If we now use Proposition 14C and Example 14.3.2, then we obtain the following table of generating functions and sequences:

generating function	sequence
$(1+X)^{-1}$	$(-1)^n$
$(1+X)^{-2}$	$(-1)^n(n+1)$
$(1+2X)^{-1}$	$(-2)^n$
$(1+2X)^{-2}$	$(-2)^n(n+1)$
$(1+2X)^{-3}$	$(-2)^n(n+2)(n+1)/2$

It follows that

$$\begin{aligned} a_n &= 3(-1)^n - 2(-1)^n(n+1) + 2(-2)^n - (-2)^n(n+1) + (-2)^n(n+2)(n+1) \\ &= (1-2n)(-1)^n + (3+2n+n^2)(-2)^n \end{aligned}$$

as before.

PROBLEMS FOR CHAPTER 16

- Solve each of the following homogeneous linear recurrences:
 - $a_{n+2} - 6a_{n+1} - 7a_n = 0$
 - $a_{n+2} + 10a_{n+1} + 25a_n = 0$
 - $a_{n+3} - 6a_{n+2} + 9a_{n+1} - 4a_n = 0$
 - $a_{n+2} - 4a_{n+1} + 8a_n = 0$
 - $a_{n+3} + 5a_{n+2} + 12a_{n+1} - 18a_n = 0$
- For each of the following linear recurrences, write down its characteristic polynomial, the general solution of the reduced recurrence, and the form of a particular solution to the recurrence:
 - $a_{n+2} + 4a_{n+1} - 5a_n = 4$
 - $a_{n+2} + 4a_{n+1} - 5a_n = n^2 + n + 1$
 - $a_{n+2} - 2a_{n+1} - 5a_n = \cos n\pi$
 - $a_{n+2} + 4a_{n+1} + 8a_n = 2^n \sin(n\pi/4)$
 - $a_{n+2} - 9a_n = 3^n$
 - $a_{n+2} - 9a_n = n3^n$
 - $a_{n+2} - 9a_n = n^2 3^n$
 - $a_{n+2} - 6a_{n+1} + 9a_n = 3^n$
 - $a_{n+2} - 6a_{n+1} + 9a_n = 3^n + 7^n$
 - $a_{n+2} + 4a_n = 2^n \cos(n\pi/2)$
 - $a_{n+2} + 4a_n = 2^n \cos n\pi$
 - $a_{n+2} + 4a_n = n2^n \sin n\pi$
- For each of the following functions $f(n)$, use the method of undetermined coefficients to find a particular solution of the non-homogeneous linear recurrence $a_{n+2} - 6a_{n+1} - 7a_n = f(n)$. Write down the general solution of the recurrence, and then find the solution that satisfies the given initial conditions:
 - $f(n) = 24(-5)^n$; $a_0 = 3$, $a_1 = -1$
 - $f(n) = 16(-1)^n$; $a_0 = 4$, $a_1 = 2$
 - $f(n) = 8((-1)^n + 7^n)$; $a_0 = 5$, $a_1 = 11$
 - $f(n) = 12n^2 - 4n + 10$; $a_0 = 0$, $a_1 = -10$
 - $f(n) = 2 \cos \frac{n\pi}{2} + 36 \sin \frac{n\pi}{2}$; $a_0 = 20$, $a_1 = 3$
- Rework Questions 3(a)–(d) using generating functions.