DISTRIBUTION OF PRIME NUMBERS

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Chapter 5

THE PRIME NUMBER THEOREM

5.1. Some Preliminary Remarks

In this chapter, we give an analytic proof of the famous Prime number theorem, a result first obtained in 1896 independently by Hadamard and de la Vallée Poussin.

THEOREM 5A. (PRIME NUMBER THEOREM) *We have*

$$\pi(X) \sim \frac{X}{\log X} \quad \textit{as } X \to \infty.$$

As in our earlier study of the distribution of primes, we use the von Mangoldt function Λ. For every $X > 0$, let

$$\psi(X) = \sum_{n \le X} \Lambda(n).$$

THEOREM 5B. *As $X \to \infty$, we have*

$$\psi(X) \sim X \qquad \textit{if and only if} \qquad \pi(X) \sim \frac{X}{\log X}.$$

Proof. Recall the proof of Theorem 2E due to Tchebycheff. We have

$$\psi(X) = \sum_{n \le X} \Lambda(n) = \sum_{\substack{p,k \\ p^k \le X}} \log p = \sum_{p \le X} (\log p) \sum_{1 \le k \le \frac{\log X}{\log p}} 1 = \sum_{p \le X} (\log p) \left[\frac{\log X}{\log p} \right] \le \pi(X) \log X. \tag{1}$$

On the other hand, for any $\alpha \in (0,1)$, we have

$$\psi(X) \ge \sum_{p \le X} \log p \ge \sum_{X^\alpha < p \le X} \log p \ge (\pi(X) - \pi(X^\alpha)) \log(X^\alpha) = \alpha(\pi(X) - \pi(X^\alpha)) \log X. \tag{2}$$

Combining (1) and (2), we have

$$\alpha \frac{\pi(X)}{X/\log X} - \alpha \frac{\pi(X^\alpha)}{X/\log X} \le \frac{\psi(X)}{X} \le \frac{\pi(X)}{X/\log X}. \tag{3}$$

Since $\alpha < 1$, it follows from Tchebycheff's theorem that

$$\frac{\pi(X^\alpha)}{X/\log X} \to 0 \quad \text{as } X \to \infty.$$

Suppose that $\pi(X) \sim X/\log X$ as $X \to \infty$. Then

$$\alpha \frac{\pi(X)}{X/\log X} - \alpha \frac{\pi(X^\alpha)}{X/\log X} \to \alpha \quad \text{as } X \to \infty.$$

It follows that for any $\epsilon > 0$, the inequality

$$\alpha - \epsilon \le \frac{\psi(X)}{X} \le 1 + \epsilon$$

holds for all large X. Since $\alpha < 1$ is arbitrary, we must have

$$\frac{\psi(X)}{X} \to 1 \quad \text{as } X \to \infty.$$

Note next that the inequalities (3) can be rewritten as

$$\frac{\psi(X)}{X} \le \frac{\pi(X)}{X/\log X} \le \frac{1}{\alpha} \frac{\psi(X)}{X} + \frac{\pi(X^\alpha)}{X/\log X}.$$

Suppose that $\psi(X) \sim X$ as $X \to \infty$. Then

$$\frac{1}{\alpha} \frac{\psi(X)}{X} + \frac{\pi(X^\alpha)}{X/\log X} \to \frac{1}{\alpha} \quad \text{as } X \to \infty.$$

It follows that for every $\epsilon > 0$, the inequality

$$1 - \epsilon \le \frac{\pi(X)}{X/\log X} \le \frac{1+\epsilon}{\alpha}$$

holds for all large X. Since $\alpha < 1$ is arbitrary, we must have

$$\frac{\pi(X)}{X/\log X} \to 1 \quad \text{as } X \to \infty.$$

This completes the proof. ○

5.2. A Smoothing Argument

To prove the Prime number theorem, it suffices to show that $\psi(X) \sim X$ as $X \to \infty$. However, a direct discussion of $\psi(X)$ introduces various tricky convergence problems. We therefore consider a smooth average of the function ψ. For $X > 0$, let

$$\psi_1(X) = \int_0^X \psi(x)\,\mathrm{d}x. \tag{4}$$

THEOREM 5C. *Suppose that $\psi_1(X) \sim \frac{1}{2}X^2$ as $X \to \infty$. Then $\psi(X) \sim X$ as $X \to \infty$.*

PROOF. Suppose that $0 < \alpha < 1 < \beta$. Since $\Lambda(n) \geq 0$ for every $n \in \mathbb{N}$, the function ψ is an increasing function. Hence for every $X > 0$, we have

$$\psi(X) \leq \frac{1}{\beta X - X}\int_X^{\beta X} \psi(x)\,\mathrm{d}x = \frac{\psi_1(\beta X) - \psi_1(X)}{(\beta - 1)X},$$

so that

$$\frac{\psi(X)}{X} \leq \frac{\psi_1(\beta X) - \psi_1(X)}{(\beta - 1)X^2}. \tag{5}$$

On the other hand, for every $X > 0$, we have

$$\psi(X) \geq \frac{1}{X - \alpha X}\int_{\alpha X}^{X} \psi(x)\,\mathrm{d}x = \frac{\psi_1(X) - \psi_1(\alpha X)}{(1 - \alpha)X},$$

so that

$$\frac{\psi(X)}{X} \geq \frac{\psi_1(X) - \psi_1(\alpha X)}{(1 - \alpha)X^2}. \tag{6}$$

As $X \to \infty$, we have

$$\frac{\psi_1(\beta X) - \psi_1(X)}{(\beta - 1)X^2} \sim \frac{1}{\beta - 1}\left(\frac{1}{2}\beta^2 - \frac{1}{2}\right) = \frac{1}{2}(\beta + 1) \tag{7}$$

and

$$\frac{\psi_1(X) - \psi_1(\alpha X)}{(1 - \alpha)X^2} \sim \frac{1}{1 - \alpha}\left(\frac{1}{2} - \frac{1}{2}\alpha^2\right) = \frac{1}{2}(\alpha + 1). \tag{8}$$

Since α and β are arbitrary, we conclude, on combining (5)–(8), that $\psi(X)/X \sim 1$ as $X \to \infty$. ○

The rest of this chapter is concerned with establishing the following crucial result.

THEOREM 5D. *We have*

$$\psi_1(X) \sim \tfrac{1}{2}X^2 \quad \text{as } X \to \infty.$$

5.3. A Contour Integral

The following result brings the Riemann zeta function $\zeta(s)$ into the argument.

THEOREM 5E. *Suppose that $X>0$ and $c>1$. Then*

$$\psi_1(X) = -\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{X^{s+1}}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s,$$

where the path of integration is the straight line $\sigma=c$.

A crucial step in the proof of Theorem 5E is provided by the following auxiliary result.

THEOREM 5F. *Suppose that $Y>0$ and $c>1$. Then*

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{Y^s}{s(s+1)}\,\mathrm{d}s = \begin{cases} 0 & \text{if } Y\le 1,\\ 1-\dfrac{1}{Y} & \text{if } Y\ge 1.\end{cases}$$

Proof. Note first of all that the integral is absolutely convergent, since

$$\left|\frac{Y^s}{s(s+1)}\right| \le \frac{Y^c}{|t|^2}$$

whenever $\sigma=c$. Let $T>1$, and write

$$I_T = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{Y^s}{s(s+1)}\,\mathrm{d}s.$$

Suppose first of all that $Y\ge 1$. Consider the circular arc $A^-(c,T)$ centred at $s=0$ and passing from $c-\mathrm{i}T$ to $c+\mathrm{i}T$ on the left of the line $\sigma=c$, and let

$$J_T^- = \frac{1}{2\pi\mathrm{i}}\int_{A^-(c,T)} \frac{Y^s}{s(s+1)}\,\mathrm{d}s.$$

Note that on $A^-(c,T)$, we have $|Y^s|=Y^\sigma\le Y^c$ since $Y\ge 1$; also we have $|s|=R$ and $|s+1|\ge R-1$, where $R=(c^2+T^2)^{1/2}$ is the radius of $A^-(c,T)$. It follows that

$$|J_T^-| \le \frac{1}{2\pi}\frac{Y^c}{R(R-1)}2\pi R \le \frac{Y^c}{T-1} \to 0 \quad \text{as } T\to\infty.$$

By Cauchy's residue theorem, we have

$$I_T = J_T^- + \operatorname{res}\left(\frac{Y^s}{s(s+1)},0\right) + \operatorname{res}\left(\frac{Y^s}{s(s+1)},-1\right) = J_T^- + 1 - \frac{1}{Y}.$$

The result for $Y\ge 1$ follows on letting $T\to\infty$.

Suppose now that $Y\le 1$. Consider the circular arc $A^+(c,T)$ centred at $s=0$ and passing from $c-\mathrm{i}T$ to $c+\mathrm{i}T$ on the right of the line $\sigma=c$, and let

$$J_T^+ = \frac{1}{2\pi\mathrm{i}}\int_{A^+(c,T)} \frac{Y^s}{s(s+1)}\,\mathrm{d}s.$$

Note that on $A^+(c,T)$, we have $|Y^s| = Y^\sigma \le Y^c$ since $Y \le 1$; also we have $|s| = R$ and $|s+1| \ge R$, where $R = (c^2+T^2)^{1/2}$ is the radius of $A^+(c,T)$. It follows that

$$|J_T^+| \le \frac{1}{2\pi}\frac{Y^c}{R^2} 2\pi R \le \frac{Y^c}{T} \to 0 \quad \text{as } T \to \infty.$$

By Cauchy's integral theorem, we have

$$I_T = J_T^+.$$

The result for $Y \le 1$ follows on letting $T \to \infty$. ◯

Proof of Theorem 5E. Note that for $X \ge 1$, we have

$$\psi_1(X) = \int_0^X \psi(x)\,\mathrm{d}x = \int_1^X \psi(x)\,\mathrm{d}x = \int_1^X \left(\sum_{n\le x} \Lambda(n)\right)\mathrm{d}x = \sum_{n\le X}(X-n)\Lambda(n),$$

the last equality following from interchanging the order of integration and summation. Note also that the above conclusion holds trivially if $0 < X < 1$. It therefore follows from Theorem 5F that for every $X > 0$, we have

$$\frac{\psi_1(X)}{X} = \sum_{n\le X}\left(1-\frac{n}{X}\right)\Lambda(n) = \sum_{n\le X}\left(1-\frac{1}{X/n}\right)\Lambda(n) = \sum_{n=1}^\infty \frac{\Lambda(n)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{(X/n)^s}{s(s+1)}\,\mathrm{d}s,$$

where $c > 1$. Since $c > 1$, the order of summation and integration can be interchanged, as

$$\sum_{n=1}^\infty \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left|\frac{\Lambda(n)(X/n)^s}{s(s+1)}\right||\mathrm{d}s| \le X^c \sum_{n=1}^\infty \frac{\Lambda(n)}{n^c}\int_{-\infty}^\infty \frac{\mathrm{d}t}{c^2+t^2}$$

is finite. It follows that

$$\frac{\psi_1(X)}{X} = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{X^s}{s(s+1)}\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s}\,\mathrm{d}s = -\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{X^s}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s$$

as required. ◯

5.4. The Riemann Zeta Function

Recall first of all Theorem 4L. In the case of the Riemann zeta function, equation (7) of Chapter 4 becomes

$$\begin{aligned}\sum_{n\le X} n^{-s} &= s\int_1^X [x]x^{-s-1}\,\mathrm{d}x + [X]X^{-s} \\ &= s\int_1^X x^{-s}\,\mathrm{d}x - s\int_1^X \{x\}x^{-s-1}\,\mathrm{d}x + X^{-s+1} - \{X\}X^{-s} \\ &= \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int_1^X \frac{\{x\}}{x^{s+1}}\,\mathrm{d}x + \frac{1}{X^{s-1}} - \frac{\{X\}}{X^s}.\end{aligned} \tag{9}$$

Letting $X \to \infty$, we deduce that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,\mathrm{d}x \tag{10}$$

if $\sigma > 1$. Recall also that (10) gives an analytic continuation of $\zeta(s)$ to $\sigma > 0$, with s simple pole at $s = 1$. We shall use these formulae to deduce important information about the order of magnitude of $|\zeta(s)|$ in the neighbourhood of the line $\sigma = 1$ and to the left of it. Note that $\zeta(\sigma + \mathrm{i}t)$ and $\zeta(\sigma - \mathrm{i}t)$ are complex conjugates, so it suffices to study $\zeta(s)$ on the upper half plane.

THEOREM 5G. *For every $\sigma \geq 1$ and $t \geq 2$, we have*
(i) $|\zeta(s)| = O(\log t)$; *and*
(ii) $|\zeta'(s)| = O(\log^2 t)$.
Suppose further that $0 < \delta < 1$. *Then for every* $\sigma \geq \delta$ *and* $t \geq 1$, *we have*
(iii) $|\zeta(s)| = O_\delta(t^{1-\delta})$.

PROOF. For $\sigma > 0$, $t \geq 1$ and $X \geq 1$, we have, by (9) and (10), that

$$\zeta(s) - \sum_{n \leq X} \frac{1}{n^s} = \frac{s}{(s-1)X^{s-1}} - \frac{1}{X^{s-1}} + \frac{\{X\}}{X^s} - s\int_X^\infty \frac{\{x\}}{x^{s+1}}\,\mathrm{d}x \tag{11}$$
$$= \frac{1}{(s-1)X^{s-1}} + \frac{\{X\}}{X^s} - s\int_X^\infty \frac{\{x\}}{x^{s+1}}\,\mathrm{d}x.$$

It follows that

$$|\zeta(s)| \leq \sum_{n \leq X} \frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + |s|\int_X^\infty \frac{\mathrm{d}x}{x^{\sigma+1}} \leq \sum_{n \leq X} \frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + \left(1 + \frac{t}{\sigma}\right)\frac{1}{X^\sigma}. \tag{12}$$

If $\sigma \geq 1$, $t \geq 1$ and $X \geq 1$, then

$$|\zeta(s)| \leq \sum_{n \leq X} \frac{1}{n} + \frac{1}{t} + \frac{1}{X} + \frac{1+t}{X} \leq (\log X + 1) + 3 + \frac{t}{X}.$$

Choosing $X = t$, we obtain

$$|\zeta(s)| \leq (\log t + 1) + 4 = O(\log t),$$

proving (i). On the other hand, if $\sigma \geq \delta$, $t \geq 1$ and $X \geq 1$, then it follows from (12) that

$$|\zeta(s)| \leq \sum_{n \leq X} \frac{1}{n^\delta} + \frac{1}{tX^{\delta-1}} + \left(2 + \frac{t}{\delta}\right)\frac{1}{X^\delta} \leq \int_0^{[X]} \frac{\mathrm{d}x}{x^\delta} + \frac{X^{1-\delta}}{t} + \frac{3t}{\delta X^\delta} \leq \frac{X^{1-\delta}}{1-\delta} + X^{1-\delta} + \frac{3t}{\delta X^\delta}.$$

Again choosing $X = t$, we obtain

$$|\zeta(s)| \leq t^{1-\delta}\left(\frac{1}{1-\delta} + 1 + \frac{3}{\delta}\right), \tag{13}$$

proving (iii). To deduce (ii), we may differentiate (11) with respect to s and proceed in a similar way. Alternatively, suppose that $s_0 = \sigma_0 + \mathrm{i}t_0$ satisfies $\sigma_0 \geq 1$ and $t_0 \geq 2$. Let C be the circle with centre s_0 and radius $\rho < 1/2$. Then Cauchy's integral formula gives

$$|\zeta'(s_0)| = \left|\frac{1}{2\pi\mathrm{i}}\int_C \frac{\zeta(s)}{(s-s_0)^2}\,\mathrm{d}s\right| \leq \frac{M}{\rho},$$

where $M = \sup_{s \in C} |\zeta(s)|$. Note next that for every $s \in C$, we clearly have $\sigma \geq \sigma_0 - \rho \geq 1 - \rho$ and $2t_0 > t \geq t_0 - \rho > 1$. It follows from (13), with $\delta = 1 - \rho$, that for every $s \in C$, we must have

$$|\zeta(s)| \leq (2t_0)^\rho\left(\frac{1}{\rho} + 1 + \frac{3}{1-\rho}\right) \leq \frac{10t_0^\rho}{\rho},$$

since $\rho < 1/2 < 1-\rho < 1$. It follows that

$$|\zeta'(s_0)| \le \frac{10t_0^\rho}{\rho^2}.$$

We now take $\rho = (\log t_0 + 2)^{-1}$. Then $t_0^\rho = \mathrm{e}^{\rho \log t_0} < \mathrm{e}$, and so

$$|\zeta'(s_0)| \le 10\mathrm{e}(\log t_0 + 2)^2.$$

(ii) now follows. ◯

THEOREM 5H. *The function $\zeta(s)$ has no zeros on the line $\sigma = 1$. Furthermore, there is a positive constant A such that as $t \to \infty$, we have, for $\sigma \ge 1$, that*

$$\frac{1}{\zeta(s)} = O\left((\log t)^A\right).$$

PROOF. For every $\theta \in \mathbb{R}$, we clearly have

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \ge 0. \tag{14}$$

On the other hand, it is easy to check that for $\sigma > 1$, we have

$$\log\zeta(s) = \sum_p \sum_{m=1}^\infty \frac{1}{mp^{ms}},$$

so that

$$\log|\zeta(\sigma+\mathrm{i}t)| = \mathfrak{Re}\left(\sum_{n=2}^\infty c_n n^{-\sigma-\mathrm{i}t}\right) = \sum_{n=2}^\infty c_n n^{-\sigma}\cos(t\log n), \tag{15}$$

where

$$c_n = \begin{cases} 1/m & \text{if } n = p^m\text{, where } p \text{ is prime and } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Combining (14)–(16), we have

$$\log|\zeta^3(\sigma)\zeta^4(\sigma+\mathrm{i}t)\zeta(\sigma+2\mathrm{i}t)| = \sum_{n=2}^\infty c_n n^{-\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) \ge 0.$$

It follows that for $\sigma > 1$, we have

$$|(\sigma-1)\zeta(\sigma)|^3 \left|\frac{\zeta(\sigma+\mathrm{i}t)}{\sigma-1}\right|^4 |\zeta(\sigma+2\mathrm{i}t)| \ge \frac{1}{\sigma-1}. \tag{17}$$

Suppose that the point $s = 1+it$ is a zero of $\zeta(s)$. Then since $\zeta(s)$ is analytic at the points $s = 1+it$ and $s = 1+2it$ and has a simple pole with residue 1 at $s = 1$, the left hand side of (17) must converge to a finite limit as $\sigma \to 1+$, contradicting the fact that the right hand side diverges to infinity as $\sigma \to 1+$. Hence $s = 1+it$ cannot be a zero of $\zeta(s)$. To prove the second assertion, we may assume without loss of generality that $1 \le \sigma \le 2$, since for $\sigma \ge 2$, we have

$$\left|\frac{1}{\zeta(s)}\right| = \left|\prod_p (1-p^{-s})\right| \le \prod_p (1+p^{-\sigma}) < \zeta(\sigma) \le \zeta(2).$$

Suppose now that $1 < \sigma \le 2$ and $t \ge 2$. Then by (17), we have

$$(\sigma-1)^3 \le |(\sigma-1)\zeta(\sigma)|^3|\zeta(\sigma+\mathrm{i}t)|^4|\zeta(\sigma+2\mathrm{i}t)| \le A_1|\zeta(\sigma+\mathrm{i}t)|^4\log(2t)$$

by Theorem 5G(i), where A_1 is a positive absolute constant. Since $\log(2t) \le 2\log t$, it follows that

$$|\zeta(\sigma+\mathrm{i}t)| \ge \frac{(\sigma-1)^{3/4}}{A_2(\log t)^{1/4}}, \tag{18}$$

where A_2 is a positive absolute constant. Note that (18) holds also when $\sigma = 1$. Suppose now that $1 < \eta < 2$. If $1 \le \sigma \le \eta$ and $t \ge 2$, then it follows from Theorem 5G(ii) that

$$|\zeta(\sigma+\mathrm{i}t)-\zeta(\eta+\mathrm{i}t)| = \left|\int_\sigma^\eta \zeta'(x+\mathrm{i}t)\,\mathrm{d}x\right| \le A_3(\eta-1)\log^2 t,$$

where A_3 is a positive absolute constant. Combining this with (18), we have

$$|\zeta(\sigma+\mathrm{i}t)| \ge |\zeta(\eta+\mathrm{i}t)| - A_3(\eta-1)\log^2 t \ge \frac{(\eta-1)^{3/4}}{A_2(\log t)^{1/4}} - A_3(\eta-1)\log^2 t. \tag{19}$$

On the other hand, if $\eta \le \sigma \le 2$ and $t \ge 2$, then in view of (18), the inequality (19) must also hold. It follows that inequality (19) holds if $1 \le \sigma \le 2$, $t \ge 2$ and $1 < \eta < 2$. We now choose η so that

$$\frac{(\eta-1)^{3/4}}{A_2(\log t)^{1/4}} = 2A_3(\eta-1)\log^2 t;$$

in other words, we choose

$$\eta = 1 + (2A_2A_3)^{-4}(\log t)^{-9},$$

where $t > t_0$, so that $\eta < 2$. Then

$$|\zeta(\sigma+\mathrm{i}t)| \ge A_3(\eta-1)\log^2 t = A_4(\log t)^{-7}$$

for $1 \le \sigma \le 2$ and $t > t_0$. ○

5.5. Completion of the Proof

We are now ready to complete the proof of Theorem 5D. By Theorem 5E, we have

$$\frac{\psi_1(X)}{X^2} = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} G(s)X^{s-1}\,\mathrm{d}s, \tag{20}$$

where $c > 1$ and $X > 0$, and where

$$G(s) = -\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s(s+1)}\zeta'(s)\frac{1}{\zeta(s)}.$$

By Theorems 4H, 5G and 5H, we know that $G(s)$ is analytic for $\sigma \ge 1$, except at $s = 1$, and that for some positive absolute constant A, we have

$$G(s) = O\left(|t|^{-2}(\log|t|)^2(\log|t|)^A\right) < |t|^{-3/2} \tag{21}$$

for all $|t| > t_0$. Let $\epsilon > 0$ be given. We now consider a contour made up of the straight line segments

$$\begin{cases} L_1 = [1 - \mathrm{i}U, 1 - \mathrm{i}T], \\ L_2 = [1 - \mathrm{i}T, \alpha - \mathrm{i}T], \\ L_3 = [\alpha - \mathrm{i}T, \alpha + \mathrm{i}T], \\ L_4 = [\alpha + \mathrm{i}T, 1 + \mathrm{i}T], \\ L_5 = [1 + \mathrm{i}T, 1 + \mathrm{i}U], \end{cases}$$

where $T = T(\epsilon) > \max\{t_0, 2\}$, $\alpha = \alpha(T) = \alpha(\epsilon) \in (0, 1)$ and U are chosen to satisfy the following three conditions:

(i) We have

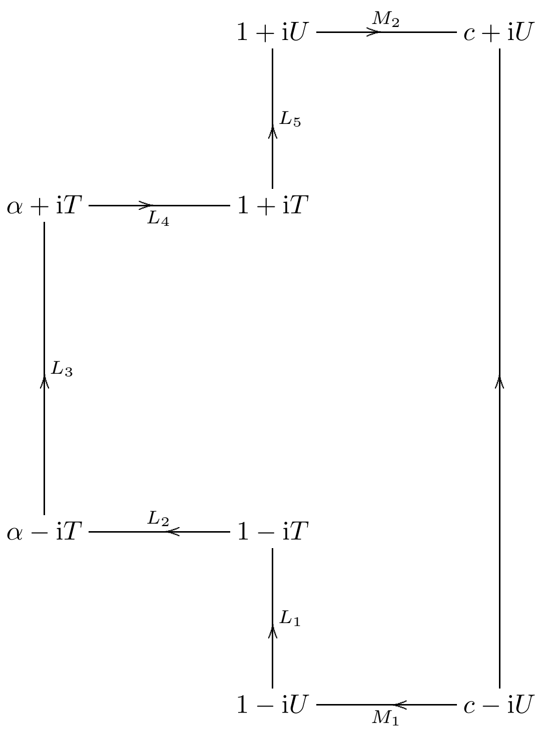
$$\int_T^\infty |G(1 + \mathrm{i}t)| \, \mathrm{d}t < \epsilon.$$

(ii) The rectangle $[\alpha, 1] \times [-T, T]$ contains no zeros of $\zeta(s)$. Note that this is possible since $\zeta(s)$ has no zeros on the line $\sigma = 1$ and, as an analytic function, has at most a finite number of zeros in the region $[1/2, 1) \times [-T, T]$.

(iii) We have $U > T$.

Furthermore, define the straight line segments

$$\begin{cases} M_1 = [c - \mathrm{i}U, 1 - \mathrm{i}U], \\ M_2 = [1 + \mathrm{i}U, c + \mathrm{i}U]. \end{cases}$$



By Cauchy's residue theorem, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}U}^{c+\mathrm{i}U} G(s) X^{s-1} \, \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{2} \int_{M_j} G(s) X^{s-1} \, \mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{5} \int_{L_j} G(s) X^{s-1} \, \mathrm{d}s + \mathrm{res}(G(s) X^{s-1}, 1), \tag{22}$$

where, for every $X > 1$, we have

$$\mathrm{res}(G(s)X^{s-1}, 1) = \tfrac{1}{2}. \tag{23}$$

Now

$$\left| \int_{L_1} G(s)X^{s-1}\,\mathrm{d}s \right| = \left| \int_{L_5} G(s)X^{s-1}\,\mathrm{d}s \right| \le \int_T^\infty |G(1+\mathrm{i}t)|\,\mathrm{d}t < \epsilon. \tag{24}$$

On the other hand,

$$\left| \int_{L_2} G(s)X^{s-1}\,\mathrm{d}s \right| = \left| \int_{L_4} G(s)X^{s-1}\,\mathrm{d}s \right| \le M \int_\alpha^1 X^{\sigma-1}\,\mathrm{d}\sigma \le \frac{M}{\log X} \tag{25}$$

and

$$\left| \int_{L_3} G(s)X^{s-1}\,\mathrm{d}s \right| \le 2TMX^{\alpha-1}, \tag{26}$$

where

$$M = M(\alpha, T) = M(\epsilon) = \sup_{L_2 \cup L_3 \cup L_4} |G(s)|. \tag{27}$$

Furthermore, by (21), we have, for $j = 1, 2$,

$$\left| \int_{M_j} G(s)X^{s-1}\,\mathrm{d}s \right| \le \int_1^c |U|^{-3/2} X^{\sigma-1}\,\mathrm{d}\sigma \le \frac{X^{c-1}}{\log X}|U|^{-3/2}. \tag{28}$$

Combining (22)–(28), we have

$$\left| \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}U}^{c+\mathrm{i}U} G(s)X^{s-1}\,\mathrm{d}s - \frac{1}{2} \right| \le \frac{\epsilon}{\pi} + \frac{M}{\pi \log X} + \frac{TM}{\pi X^{1-\alpha}} + \frac{X^{c-1}|U|^{-3/2}}{\pi \log X}.$$

On letting $U \to \infty$, we have

$$\left| \frac{\psi_1(X)}{X^2} - \frac{1}{2} \right| \le \frac{\epsilon}{\pi} + \frac{M}{\pi \log X} + \frac{TM}{\pi X^{1-\alpha}}.$$

It then follows that

$$\lim_{X \to \infty} \left| \frac{\psi_1(X)}{X^2} - \frac{1}{2} \right| \le \frac{\epsilon}{\pi}.$$

Note finally that $\epsilon > 0$ is arbitrary, and the left hand side is independent of ϵ. It follows that

$$\lim_{X \to \infty} \frac{\psi_1(X)}{X^2} = \frac{1}{2}.$$

This completes the proof of Theorem 5D.