DISTRIBUTION OF PRIME NUMBERS

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Chapter 6

THE RIEMANN ZETA FUNCTION

6.1. Riemann's Memoir

In Riemann's only paper on number theory, published in 1860, he proved the following result.

THEOREM 6A. (RIEMANN) *The function $\zeta(s)$ can be continued analytically over the whole complex plane $\mathbb{C}$, and satisfies the functional equation*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1}$$

where Γ denotes the gamma function. In particular, the function $\zeta(s)$ is analytic everywhere, except for a simple pole at $s=1$ with residue 1.

Note that the functional equation (1) enables properties of $\zeta(s)$ for $\sigma<0$ to be inferred from properties of $\zeta(s)$ for $\sigma>1$.

Remarks. (i) As can be observed from the functional equation (1), the study of the Riemann zeta function depends intimately on properties of the gamma function. The latter is usually defined by the Euler integral

$$\Gamma(s)=\int_0^\infty \mathrm{e}^{-t}t^{s-1}\,\mathrm{d}t,$$

valid whenever $\mathfrak{Re}\,s>0$, and satisfies $\Gamma(s+1)=s\Gamma(s)$. The Weierstrass formula

$$\frac{1}{s\Gamma(s)}=\mathrm{e}^{\gamma s}\prod_{n=1}^\infty\left(1+\frac{s}{n}\right)\mathrm{e}^{-s/n},$$

where γ is Euler's constant, extends the gamma function to the whole complex plane $\mathbb{C}$. It is then easy to see that $\Gamma(s)$ has no zeros, but has simple poles at $s = 0, -1, -2, \ldots$.

(ii) The formulas

$$\Gamma\left(\frac{s}{2}\right) = \pi^{-1/2} s^{1-s} \Gamma(s) \Gamma\left(\frac{1-s}{2}\right) \cos\frac{\pi s}{2}$$

and

$$\Gamma\left(\frac{1-s}{2}\right) = \pi^{-1/2} 2^{s} \Gamma(1-s) \Gamma\left(\frac{s}{2}\right) \cos\frac{\pi(1-s)}{2}$$

are particularly useful in the study of $\zeta(s)$, as we shall see later in the proof of Theorem 6V.

(iii) Stirling's asymptotic formula

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{\log 2\pi}{2} + O(|s|^{-1}) \quad \text{as } |s| \to \infty$$

is valid in any angle $-\pi + \delta < \arg s < \pi - \delta$ for any fixed $\delta > 0$. The same condition gives the estimate

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1}) \quad \text{as } |s| \to \infty.$$

(iv) The interested reader may refer to Chapters 12 and 13 in the volume *Modern Analysis* by Whittaker and Watson for detailed proofs of the above.

In view of Remark (i) above, the only zeros of $\zeta(s)$ for $\sigma < 0$ are at the poles of $\Gamma(s/2)$; in other words, at the points $s = -2, -4, -6, \ldots$. These are called the trivial zeros of $\zeta(s)$.

The part of the plane with $0 \le \sigma \le 1$ is called the critical strip.

Riemann's paper is particularly remarkable for the conjectures it contains. While most of these conjectures have been proved, the famous Riemann hypothesis has so far resisted all attempts to prove or disprove it.

THEOREM 6B. (HADAMARD 1893) *The function $\zeta(s)$ has infinitely many zeros in the critical strip.*

It is easy to see that the zeros of $\zeta(s)$ in the critical strip are placed symmetrically with respect to the line $t = 0$ as well as with respect to the line $\sigma = 1/2$, the latter observation being a consequence of the functional equation (1).

THEOREM 6C. (HADAMARD 1893) *The entire function*

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{2}$$

has the product representation

$$\xi(s) = \mathrm{e}^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \mathrm{e}^{s/\rho}, \tag{3}$$

where A and B are constants and where ρ runs over all the zeros of the function $\zeta(s)$ in the critical strip.

We comment here that the product representation (3) plays an important role in the first proof of the Prime number theorem.

THEOREM 6D. (VON MANGOLDT 1905) *Let* $N(T)$ *denote the number of zeros* $\rho = \beta + \mathrm{i}\gamma$ *of the function* $\zeta(s)$ *in the critical strip with* $0 < \gamma \le T$. *Then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{4}$$

The most remarkable of Riemann's conjectures is an explicit formula for the difference $\pi(X) - \mathrm{li}(X)$, containing a term which is a sum over the zeros of $\zeta(s)$ in the critical strip. This shows that the zeros of $\zeta(s)$ plays a crucial role in the study of the distribution of primes. Here we state a result closely related to this formula.

THEOREM 6E. (VON MANGOLDT 1895) *Let*

$$\psi(X) = \sum_{n \le X} \Lambda(n) \qquad \text{and} \qquad \psi_0(X) = \frac{\psi(X-0) + \psi(X+0)}{2}.$$

Then

$$\psi_0(X) - X = -\sum_{\rho} \frac{X^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{X^2}\right),$$

where the terms in the sum arising from complex conjugates are taken together.

However, there remains one of Riemann's conjectures which is still unsolved today. The open question below is arguably the most famous unsolved problem in the whole of mathematics.

CONJECTURE. (RIEMANN HYPOTHESIS) *The zeros of the function* $\zeta(s)$ *in the critical strip all lie on the line* $\sigma = 1/2$.

We shall nevertheless establish the following rather weak partial result which gives a zero-free region for $\zeta(s)$. This will be sufficient to give another proof of the Prime number theorem, via the explicit formula given in Theorem 6E.

THEOREM 6F. (DE LA VALLÉE-POUSSIN 1899) *There exists an absolute constant* $c > 0$ *such that the function* $\zeta(s)$ *has no zeros in the region*

$$\sigma \ge 1 - \frac{c}{\log t} \qquad \text{and} \qquad t \ge 2.$$

6.2. Riemann's Proof of the Functional Equation

Suppose that $\sigma > 0$. Writing $t = n^2\pi x$, we have

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{s/2-1}\mathrm{e}^{-t}\,\mathrm{d}t = (n^2\pi)^{s/2}\int_0^\infty x^{s/2-1}\mathrm{e}^{-n^2\pi x}\,\mathrm{d}x,$$

so that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty x^{s/2-1}\mathrm{e}^{-n^2\pi x}\,\mathrm{d}x.$$

It follows that for $\sigma > 1$, we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{s/2-1}\mathrm{e}^{-n^2\pi x}\,\mathrm{d}x = \int_0^{\infty} x^{s/2-1}\left(\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}\right)\mathrm{d}x,$$

where the change of order of summation and integration is justified by the convergence of

$$\sum_{n=1}^{\infty}\int_0^{\infty} x^{\sigma/2-1}\mathrm{e}^{-n^2\pi x}\,\mathrm{d}x.$$

Now write

$$\omega(x) = \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}.$$

Then for $\sigma > 1$, we have

$$\begin{aligned} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_1^{\infty} x^{s/2-1}\omega(x)\,\mathrm{d}x + \int_0^{1} y^{s/2-1}\omega(y)\,\mathrm{d}y \\ &= \int_1^{\infty} x^{s/2-1}\omega(x)\,\mathrm{d}x + \int_1^{\infty} x^{-s/2-1}\omega(x^{-1})\,\mathrm{d}x. \end{aligned} \tag{5}$$

We shall show that for every $x > 0$, the function

$$\theta(x) = \sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi x} = 1 + 2\omega(x)$$

satisfies the functional equation $\theta(x^{-1}) = x^{1/2}\theta(x)$ which can be written in the form

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi/x} = x^{1/2}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi x}. \tag{6}$$

It then follows that

$$2\omega(x^{-1}) = \theta(x^{-1}) - 1 = x^{1/2}\theta(x) - 1 = -1 + x^{1/2} + 2x^{1/2}\omega(x),$$

so that for $\sigma > 1$, we have

$$\begin{aligned} \int_1^{\infty} x^{-s/2-1}\omega(x^{-1})\,\mathrm{d}x &= \int_1^{\infty} x^{-s/2-1}\left(-\frac{1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}\omega(x)\right)\mathrm{d}x \\ &= \frac{1}{s(s-1)} + \int_1^{\infty} x^{-s/2-1/2}\omega(x)\,\mathrm{d}x. \end{aligned} \tag{7}$$

It follows on combining (5) and (7) that for $\sigma > 1$, we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty}\left(x^{s/2-1} + x^{-s/2-1/2}\right)\omega(x)\,\mathrm{d}x. \tag{8}$$

Note now that the integral on the right hand side of (8) converges absolutely for any s, and uniformly in any bounded part of the plane, since $\omega(x) = O(\mathrm{e}^{-\pi x})$ as $x \to +\infty$. Hence the integral represents an entire function of s, and the formula gives the analytic continuation of $\zeta(s)$ over the whole plane. Note also that the right hand side of (8) remains unchanged when s is replaced by $1-s$, so that the functional equation (1) follows immediately. Finally, note that the function

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is analytic everywhere. Since $s\Gamma(s/2)$ has no zeros, the only possible pole of $\zeta(s)$ is at $s=1$, and we have already shown earlier that $\zeta(s)$ has a simple pole at $s=1$ with residue 1.

It remains to establish the functional equation (6) for every $x>0$. The starting point is the Poisson summation formula, that under certain conditions on a function $f(t)$, we have

$$\sideset{}{'}\sum_{A\le n\le B} f(n)=\sum_{\nu=-\infty}^{\infty}\int_A^B f(t)\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t, \tag{9}$$

where $\sum'$ denotes that the terms in the sum corresponding to $n=A$ and $n=B$ are $\frac{1}{2}f(A)$ and $\frac{1}{2}f(B)$ respectively. Using (9) with $A=-N$, $B=N$ and $f(t)=\mathrm{e}^{-t^2\pi/x}$, we have

$$\sideset{}{'}\sum_{n=-N}^{N}\mathrm{e}^{-n^2\pi/x}=\sum_{\nu=-\infty}^{\infty}\int_{-N}^{N}\mathrm{e}^{-t^2\pi/x}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t.$$

Letting $N\to\infty$, we obtain

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi/x}=\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-t^2\pi/x}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t. \tag{10}$$

This is justified by noting that

$$\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)\mathrm{e}^{-t^2\pi/x}\mathrm{e}^{2\pi\mathrm{i}\nu t}\,\mathrm{d}t=2\int_N^{\infty}\mathrm{e}^{-t^2\pi/x}\cos(2\pi\nu t)\,\mathrm{d}t,$$

and that

$$\left|\sum_{\nu\ne0}\int_N^{\infty}\mathrm{e}^{-t^2\pi/x}\cos(2\pi\nu t)\,\mathrm{d}t\right|\to0\quad\text{as } N\to\infty.$$

Writing $t=xu$ and using (10), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi/x}&=x\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2\pi x}\mathrm{e}^{2\pi\mathrm{i}\nu xu}\,\mathrm{d}u=x\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-(u-\mathrm{i}\nu)^2\pi x-\nu^2\pi x}\,\mathrm{d}u\\&=x\sum_{\nu=-\infty}^{\infty}\mathrm{e}^{-\nu^2\pi x}\int_{-\infty}^{\infty}\mathrm{e}^{-(u-\mathrm{i}\nu)^2\pi x}\,\mathrm{d}u.\end{aligned} \tag{11}$$

Note now that the function $\mathrm{e}^{-z^2\pi x}$ is an entire function of the complex variable z. It follows from Cauchy's integral theorem that

$$\int_{-\infty}^{\infty}\mathrm{e}^{-(u-\mathrm{i}\nu)^2\pi x}\,\mathrm{d}u=\int_{-\infty}^{\infty}\mathrm{e}^{-u^2\pi x}\,\mathrm{d}u=Ax^{-1/2}, \tag{12}$$

where

$$A=\int_{-\infty}^{\infty}\mathrm{e}^{-y^2\pi}\,\mathrm{d}y=1. \tag{13}$$

The functional equation (6) now follows on combining (11)–(13), and the proof of Theorem 6A is now complete.

6.3. Entire Functions

In this section, we shall prove some technical results on entire functions for use later in the proof of Theorems 6B and 6C.

An entire function $f(s)$ is said to be of order 1 if

$$f(s) = O_\alpha\left(\mathrm{e}^{|s|^\alpha}\right) \quad \text{as } |s| \to \infty \tag{14}$$

holds for every $\alpha > 1$ and fails for every $\alpha < 1$.

Suppose that the entire function $h(s)$ has no zeros on the plane. Then the function $g(s) = \log h(s)$ can be defined as a single valued function and is also entire. Suppose that

$$h(s) = O_\alpha\left(\mathrm{e}^{|s|^\alpha}\right) \quad \text{as } |s| \to \infty \tag{15}$$

holds for every $\alpha > 1$. Then

$$\mathfrak{Re}\, g(R\mathrm{e}^{\mathrm{i}\theta}) = \log |h(R\mathrm{e}^{\mathrm{i}\theta})| = O_\alpha(R^\alpha) \quad \text{as } R \to \infty$$

holds for every $\alpha > 1$. Without loss of generality, we may suppose that $g(0) = 0$. Then we can write

$$g(R\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=1}^\infty (a_k + \mathrm{i}b_k) R^k \mathrm{e}^{\mathrm{i}k\theta}, \quad \text{where } a_k, b_k \in \mathbb{R},$$

so that

$$\mathfrak{Re}\, g(R\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=1}^\infty a_k R^k \cos k\theta - \sum_{k=1}^\infty b_k R^k \sin k\theta.$$

Note now that for every $k, n \in \mathbb{N}$, we have

$$\int_0^{2\pi} \cos k\theta \cos n\theta \, \mathrm{d}\theta = \begin{cases} \pi & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$$

and

$$\int_0^{2\pi} \sin k\theta \cos n\theta \, \mathrm{d}\theta = 0.$$

It follows that

$$\int_0^{2\pi} \left(\mathfrak{Re}\, g(R\mathrm{e}^{\mathrm{i}\theta})\right) \cos n\theta \, \mathrm{d}\theta = \pi a_n R^n,$$

so that

$$\pi |a_n| R^n \leq \int_0^{2\pi} \left|\mathfrak{Re}\, g(R\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta = O_\alpha(R^\alpha)$$

holds for all sufficiently large R and every $\alpha > 1$. On letting $R \to \infty$, we see that $a_n = 0$ for every $n > 1$. A similar argument using the function $\sin n\theta$ instead of the function $\cos n\theta$ gives $b_n = 0$ for every $n > 1$. We have therefore proved the following result.

THEOREM 6G. *Suppose that the entire function $h(s)$ has no zeros on the complex plane $\mathbb{C}$, and that* (15) *holds for every $\alpha > 1$. Then $h(s) = \mathrm{e}^{A+Bs}$, where A and B are constants.*

Remark. In the preceding argument, note that it is enough to assume that the estimates for $h(s)$ hold for a sequence of values R with limit infinity.

Our next task is to study the distribution of the zeros of an entire function. The first step in this direction is summarized by the result below.

THEOREM 6H. (JENSEN'S FORMULA) *Suppose that an entire function $f(s)$ satisfies $f(0) \neq 0$. Suppose further that $s_1, \ldots, s_n$ are the zeros of $f(s)$ in $|s| < R$, counted with multiplicities, and that there are no zeros of $f(s)$ on $|s| = R$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(R\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta - \log|f(0)| = \log\frac{R^n}{|s_1 \ldots s_n|}. \tag{16}$$

Proof. We may clearly write

$$f(s) = (s - s_1) \ldots (s - s_n)k(s),$$

where $k(s)$ is analytic and has no zeros in $|s| \le R$, so that $\log k(s)$ is analytic in $|s| \le R$. It follows from Gauss's mean value theorem that

$$\frac{1}{2\pi}\int_0^{2\pi} \log k(R\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \log k(0).$$

Taking real parts, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log|k(R\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta = \log|k(0)| = \log|f(0)| - \log|s_1 \ldots s_n|. \tag{17}$$

Unfortunately, for every $j = 1, \ldots, n$, we cannot apply a similar argument to $\log|s - s_j|$, since the function $s - s_j$ has a zero at s_j. Note, however, that the function

$$\frac{R^2 - \overline{s_j}s}{R}$$

has no zeros in $|s| \le R$ and satisfies

$$\left|\frac{R^2 - \overline{s_j}s}{R}\right| = |s - s_j|$$

on the circle $|s| = R$, so that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|R\mathrm{e}^{\mathrm{i}\theta} - s_j|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{R^2 - \overline{s_j}R\mathrm{e}^{\mathrm{i}\theta}}{R}\right|\,\mathrm{d}\theta. \tag{18}$$

Clearly the function

$$\log\frac{R^2 - \overline{s_j}s}{R}$$

is analytic in $|s| \leq R$. Applying Gauss's mean value theorem over the circle $|s| = R$ on this function and taking real parts, we conclude that the right hand side of (18) is equal to $\log R$. Finally, note that

$$\log |f(R\mathrm{e}^{\mathrm{i}\theta})| = \sum_{j=1}^{n} \log |R\mathrm{e}^{\mathrm{i}\theta} - s_j| + \log |k(R\mathrm{e}^{\mathrm{i}\theta})|,$$

so that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(R\mathrm{e}^{\mathrm{i}\theta})| \,\mathrm{d}\theta = n \log R + \log |f(0)| - \log |s_1 \ldots s_n|.$$

This completes the proof. ○

Remarks. (i) It is important to point out that Jensen's formula was in fact only discovered after Hadamard's work in connection with Theorems 6B and 6C.

(ii) Gauss's mean value theorem states that the value of an analytic function at the centre of a circle is equal to the arithmetic mean of its values on the circle. In particular, if the function $F(s)$ is analytic for $|s| < R_0$, then for every $R < R_0$, we have

$$F(0) = \frac{1}{2\pi} \int_0^{2\pi} F(R\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta.$$

A simple consequence of Jensen's formula is the following result on the zeros of entire functions.

THEOREM 6J. *Suppose that $f(s)$ is an entire function satisfying $f(0) \neq 0$, and that* (14) *holds for every $\alpha > 1$. Suppose further that $s_1, s_2, s_3, \ldots$ are the zeros of $f(s)$, counted with multiplicities and where $|s_1| \leq |s_2| \leq |s_3| \leq \ldots$. Then for every $\alpha > 1$, the series*

$$\sum_{n=1}^{\infty} |s_n|^{-\alpha}$$

is convergent.

Proof. Note that the right hand side of (16) is equal to

$$\int_0^R r^{-1} n(r) \,\mathrm{d}r,$$

where, for every non-negative $r \leq R$, $n(r)$ denotes the number of zeros of $f(s)$ in $|s| \leq r$. To see this, note that

$$\begin{aligned}\int_0^R r^{-1} n(r) \,\mathrm{d}r &= \sum_{j=1}^{n-1} \int_{|s_j|}^{|s_{j+1}|} r^{-1} j \,\mathrm{d}r + \int_{|s_n|}^{R} r^{-1} n \,\mathrm{d}r \\ &= \sum_{j=1}^{n-1} j(\log |s_{j+1}| - \log |s_j|) + n(\log R - \log |s_n|) \\ &= n \log R - \log |s_1| - \ldots - \log |s_n|.\end{aligned}$$

For every $\alpha > 1$, write $\alpha^* = (\alpha + 1)/2$, so that $1 < \alpha^* < \alpha$. Then

$$\log |f(R\mathrm{e}^{\mathrm{i}\theta})| = O_\alpha(R^{\alpha^*}) \quad \text{as } R \to \infty,$$

so that by Jensen's formula, we have

$$\int_0^R r^{-1}n(r)\,\mathrm{d}r = O_\alpha(R^{\alpha^*}) - \log|f(0)| = O_\alpha(R^{\alpha^*}) \quad \text{as } R\to\infty.$$

On the other hand, note that

$$\int_R^{2R} r^{-1}n(r)\,\mathrm{d}r \ge n(R)\int_R^{2R} r^{-1}\,\mathrm{d}r = n(R)\log 2.$$

It follows that

$$n(R) = O_\alpha(R^{\alpha^*}) \quad \text{as } R\to\infty.$$

Hence

$$\sum_{n=1}^\infty |s_n|^{-\alpha} = \int_0^\infty r^{-\alpha}\,\mathrm{d}n(r) = \alpha\int_0^\infty r^{-\alpha-1}n(r)\,\mathrm{d}r < \infty.$$

This completes the proof. ○

Suppose now that $f(s)$ is an entire function satisfying $f(0)\ne 0$, and that (14) holds for every $\alpha>1$. Suppose further that $s_1,s_2,s_3,\ldots$ are the zeros of $f(s)$, counted with multiplicities and where $|s_1|\le|s_2|\le|s_3|\le\ldots$. Then for every $\epsilon>0$, the series

$$\sum_{n=1}^\infty |s_n|^{-1-\epsilon}$$

converges, so that the series

$$\sum_{n=1}^\infty |s_n|^{-2}$$

converges, and so the product

$$P(s) = \prod_{n=1}^\infty \left(1-\frac{s}{s_n}\right)\mathrm{e}^{s/s_n} \tag{19}$$

converges absolutely for every $s\in\mathbb{C}$, and uniformly in any bounded domain not containing any zeros of $f(s)$. It follows that $P(s)$ is an entire function, with zeros at $s_1,s_2,s_3,\ldots$. Now write

$$f(s) = P(s)h(s), \tag{20}$$

where $h(s)$ is an entire function without zeros. If (15) holds for every $\alpha>1$, then $h(s)=\mathrm{e}^{A+Bs}$, where A and B are constants, and so

$$f(s) = \mathrm{e}^{A+Bs}\prod_{n=1}^\infty \left(1-\frac{s}{s_n}\right)\mathrm{e}^{s/s_n}. \tag{21}$$

THEOREM 6K. *Under the hypotheses of Theorem 6J, the inequality* (15) *holds for every* $\alpha>1$, *where the function* $h(s)$ *is defined by* (19) *and* (20). *In particular, the function* $f(s)$ *can be expressed in the form* (21), *where* A *and* B *are constants.*

Proof. To show that the inequality (15) holds for every $\alpha > 1$, it clearly suffices, in view of (14) and (20), to establish a suitable lower bound for $|P(s)|$. Since the series

$$\sum_{n=1}^\infty |s_n|^{-2}$$

is convergent, the set

$$\mathcal{S} = \bigcup_{n=1}^\infty (|s_n| - |s_n|^{-2}, |s_n| + |s_n|^{-2})$$

has finite total length. It follows that there exist arbitrarily large positive real numbers R such that $R \notin \mathcal{S}$. It is easy to see that for any such real number $R \notin \mathcal{S}$, we have

$$|R - |s_n|| \ge |s_n|^{-2} \quad \text{for every } n \in \mathbb{N}. \tag{22}$$

The idea now is to split up the product $P(s)$ into three products according to the size of $n \in \mathbb{N}$ relative to R. More precisely, for any such $R \notin \mathcal{S}$, write

$$P(s) = P_1(s)P_2(s)P_3(s), \tag{23}$$

where for every $j = 1, 2, 3$, we have

$$P_j(s) = \prod_{(24.j)} \left(1 - \frac{s}{s_n}\right) \mathrm{e}^{s/s_n},$$

where the products are taken over all $n \in \mathbb{N}$ satisfying

$$|s_n| < \frac{R}{2}, \tag{24.1}$$

$$\frac{R}{2} \le |s_n| < 2R, \tag{24.2}$$

$$|s_n| \ge 2R, \tag{24.3}$$

respectively. Let $\epsilon > 0$ be chosen and fixed.

Suppose first of all that (24.1) holds. Then on $|s| = R$, we have

$$\left|\left(1 - \frac{s}{s_n}\right) \mathrm{e}^{s/s_n}\right| \ge \left(\left|\frac{s}{s_n}\right| - 1\right) \mathrm{e}^{-|s|/|s_n|} > \mathrm{e}^{-R/|s_n|},$$

and so it follows from

$$\sum_{(24.1)} |s_n|^{-1} < \left(\frac{R}{2}\right)^\epsilon \sum_{n=1}^\infty |s_n|^{-1-\epsilon}$$

that

$$|P_1(s)| \gg_\epsilon \mathrm{e}^{-R^{1+2\epsilon}} \quad \text{as } R \to \infty. \tag{25}$$

Suppose next that (24.2) holds. Then on $|s| = R$, we have

$$\left|\left(1 - \frac{s}{s_n}\right) \mathrm{e}^{s/s_n}\right| \ge \left|\frac{s_n - s}{s_n}\right| \mathrm{e}^{-|s|/|s_n|} > \frac{||s_n| - R|}{2R} \mathrm{e}^{-2} \gg R^{-3},$$

in view of (22). Note that there are at most $O_\epsilon(R^{1+\epsilon})$ values of n for which (24.2) holds. Hence on $|s| = R$, we have

$$|P_2(s)| \gg_\epsilon (R^{-3})^{R^{1+\epsilon}} \gg_\epsilon \mathrm{e}^{-R^{1+2\epsilon}} \quad \text{as } R \to \infty. \tag{26}$$

Suppose finally that (24.3) holds. Then on $|s| = R$, we have

$$\left|\left(1 - \frac{s}{s_n}\right)\mathrm{e}^{s/s_n}\right| > \mathrm{e}^{-c(R/|s_n|)^2} \tag{27}$$

for some positive constant c (see the Remark below), and so it follows from

$$\sum_{(24.3)} |s_n|^{-2} \le (2R)^{-1+\epsilon} \sum_{n=1}^{\infty} |s_n|^{-1-\epsilon}$$

that

$$|P_3(s)| \gg_\epsilon \mathrm{e}^{-R^{1+2\epsilon}} \quad \text{as } R \to \infty. \tag{28}$$

It now follows from (23), (25), (26) and (28) that on $|s| = R$, we have

$$|P(s)| \gg_\epsilon \mathrm{e}^{-R^{1+3\epsilon}} \quad \text{as } R \to \infty. \tag{29}$$

The result then follows on combining (20) and (29), and noting that the inequality (14) holds for $\alpha = 1+\epsilon$. ○

REMARK. Note that the inequality (27) is of the form

$$|(1 - z)\mathrm{e}^z| > \mathrm{e}^{-c|z|^2}, \tag{30}$$

where $|z| \le 1/2$. Write $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. Then (30) will follow if we show that

$$(1 - x)^2\mathrm{e}^{2x} > \mathrm{e}^{-2cx^2}$$

whenever $|x| \le 1/2$. This last inequality can easily be established by using the theory of real valued functions of a real variable.

Finally, we make the following simple observation.

THEOREM 6L. *Under the hypotheses of Theorem 6J, suppose further that the series*

$$\sum_{n=1}^{\infty} |s_n|^{-1}$$

is convergent. Then there exists a positive constant c such that

$$f(s) = O(\mathrm{e}^{c|s|}) \quad \text{as } |s| \to \infty.$$

PROOF. This follows from (21) and the inequality $|(1 - z)\mathrm{e}^z| \le \mathrm{e}^{2|z|}$ which holds for every $z \in \mathbb{C}$. ○

6.4. Zeros of the Zeta Function

Recall that the function $\xi(s)$, defined by (2), is an entire function, and that $\xi(0) \neq 0$. Note also that the zeros of $\xi(s)$ are precisely the zeros of $\zeta(s)$ in the critical strip. In order to establish Theorem 6C, we shall use Theorem 6K. We therefore first need to show that for every $\alpha > 1$, we have

$$\xi(s) = O_\alpha\left(\mathrm{e}^{|s|^\alpha}\right) \quad \text{as } |s| \to \infty.$$

We shall in fact prove the following stronger result.

THEOREM 6M. *There exists a positive constant c such that*

$$|\xi(s)| < \mathrm{e}^{c|s|\log|s|} \quad \textit{as } |s| \to \infty. \tag{31}$$

Furthermore, for any positive constant c, the inequality

$$|\xi(s)| < \mathrm{e}^{c|s|} \quad \textit{as } |s| \to \infty \tag{32}$$

does not hold.

Proof. Since $\xi(s) = \xi(1-s)$ for every $s \in \mathbb{C}$, it suffices to prove the inequality (31) for $\sigma \geq 1/2$. First of all, there exists a positive constant c_1 such that

$$\left|\frac{1}{2}s(s-1)\pi^{-s/2}\right| < \mathrm{e}^{c_1|s|}.$$

Next, Stirling's formula

$$\log\Gamma\left(\frac{s}{2}\right) = \left(\frac{s}{2} - \frac{1}{2}\right)\log\frac{s}{2} - \frac{s}{2} + \frac{1}{2}\log 2\pi + O(|s|^{-1})$$

as $|s| \to \infty$ is valid in the angle $-\pi/2 < \arg s < \pi/2$, and so there exists a positive constant c_2 such that

$$\left|\Gamma\left(\frac{s}{2}\right)\right| < \mathrm{e}^{c_2|s|\log|s|}.$$

Finally, note that the formula

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}\,\mathrm{d}x$$

is valid for $\sigma > 0$, and the integral is bounded for $\sigma \geq 1/2$, so that there exists a positive constant c_3 such that

$$|\zeta(s)| < c_3|s|.$$

This proves (31). On the other hand, note that as $s \to +\infty$ through real values, we have

$$\log\Gamma\left(\frac{s}{2}\right) \sim \frac{s}{2}\log\frac{s}{2} \qquad \text{and} \qquad \zeta(s) \to 1,$$

so that (32) does not hold. ○

To complete the proof of Theorems 6B and 6C, note that by Theorem 6L, the series

$$\sum_\rho |\rho|^{-1}$$

is divergent, where ρ denotes the zeros of $\xi(s)$ and so the zeros of $\zeta(s)$ in the critical strip. Theorem 6B follows immediately. Theorem 6C now follows from Theorems 6K and 6M.

6.5. An Important Formula

It follows from (3) that

$$\log\xi(s) = A + Bs + \sum_\rho \left(\frac{s}{\rho} + \log\left(1 - \frac{s}{\rho}\right)\right).$$

Differentiating with respect to s, we obtain

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho \left(\frac{1}{\rho} + \frac{1}{s-\rho}\right). \tag{33}$$

On the other hand, it follows from (2) and $s\Gamma(s) = \Gamma(s+1)$ that

$$\log\xi(s) = \log(s-1) - \frac{s}{2}\log\pi + \log\Gamma\left(\frac{s}{2}+1\right) + \log\zeta(s).$$

Differentiating with respect to s, we obtain

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \frac{\zeta'(s)}{\zeta(s)}. \tag{34}$$

Combining (33) and (34), we obtain the following result.

THEOREM 6N. *We have*

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_\rho \left(\frac{1}{\rho} + \frac{1}{s-\rho}\right), \tag{35}$$

where B is a constant and where ρ denotes the zeros of the function $\zeta(s)$ in the critical strip.

The formula (35) clearly exhibits the pole of $\zeta(s)$ at $s = 1$ and the zeros ρ in the critical strip. On the other hand, the trivial zeros are exhibited by the term

$$-\frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)}.$$

To see this last point, we start from the Weierstrass formula

$$\frac{1}{\Gamma(s+1)} = \frac{1}{s\Gamma(s)} = \mathrm{e}^{\gamma s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)\mathrm{e}^{-s/n},$$

where γ is Euler's constant. This gives

$$\frac{1}{\Gamma(\frac{s}{2}+1)} = \mathrm{e}^{\gamma s/2}\prod_{n=1}^\infty \left(1 + \frac{s}{2n}\right)\mathrm{e}^{-s/2n}.$$

Taking logarithms, we obtain

$$-\log\Gamma\left(\frac{s}{2}+1\right) = \frac{1}{2}\gamma s + \sum_{n=1}^\infty \left(\log\left(1 + \frac{s}{2n}\right) - \frac{s}{2n}\right).$$

Differentiating with respect to s, we obtain

$$-\frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} = \frac{1}{2}\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right).$$

6.6. A Zero-Free Region

Recall Theorem 5H, where we show that the function $\zeta(s)$ has no zeros on the line $\sigma = 1$ by using the function $\log\zeta(s)$ together with the observation that

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0.$$

Here it is more convenient to work with the logarithmic derivative $\zeta'(s)/\zeta(s)$, since its only poles for $\sigma > 0$ are at $s = 1$ and the zeros of $\zeta(s)$ in the critical strip. Starting from the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

valid for $\sigma > 1$, we immediately deduce that

$$\mathfrak{Re}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\cos(t\log n).$$

It follows that for every $\sigma > 1$,

$$3\left(-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) + 4\mathfrak{Re}\left(-\frac{\zeta'(\sigma+\mathrm{i}t)}{\zeta(\sigma+\mathrm{i}t)}\right) + \mathfrak{Re}\left(-\frac{\zeta'(\sigma+2\mathrm{i}t)}{\zeta(\sigma+2\mathrm{i}t)}\right) \geq 0. \tag{36}$$

The simple pole of $\zeta(s)$ at $s = 1$ leads to a simple pole of $-\zeta'(s)/\zeta(s)$ there with residue 1. Hence there exists a positive absolute constant A_1 such that

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma-1} + A_1 \quad \text{if } 1 < \sigma \leq 2. \tag{37}$$

On the other hand, it is well known that there exists a positive absolute constant A_2 such that the gamma function $\Gamma(s)$ satisfies the inequality

$$\frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} < A_2\log t \quad \text{if } 1 < \sigma \leq 2 \text{ and } t \geq 2.$$

It follows from the identity (35) that there exists a positive absolute constant A_3 such that

$$\mathfrak{Re}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) < A_3\log t - \sum_{\rho}\mathfrak{Re}\left(\frac{1}{\rho} + \frac{1}{s-\rho}\right) \quad \text{if } 1 < \sigma \leq 2 \text{ and } t \geq 2. \tag{38}$$

Suppose that $\rho = \beta + \mathrm{i}\gamma$, where $\beta, \gamma \in \mathbb{R}$, is a zero of the function $\zeta(s)$ in the critical strip. Then $0 < \beta < 1$, and since $\sigma > 1$, we have

$$\mathfrak{Re}\frac{1}{\rho} = \frac{\beta}{|\rho|^2} > 0 \qquad \text{and} \qquad \mathfrak{Re}\frac{1}{s-\rho} = \frac{\sigma-\beta}{|s-\rho|^2} > 0.$$

This means that the inequality (38) remains valid if we omit any term from the sum on the right hand side. In particular, when $s = \sigma + 2\mathrm{i}t$, we have the inequality

$$\mathfrak{Re}\left(-\frac{\zeta'(\sigma+2\mathrm{i}t)}{\zeta(\sigma+2\mathrm{i}t)}\right) < A_3\log(2t) < A_4\log t \quad \text{if } 1<\sigma\le 2 \text{ and } t\ge 2, \tag{39}$$

where A_4 is a positive absolute constant.

Suppose now that $t \geq 2$ is fixed and there exists a real number β such that $\rho = \beta + \mathrm{i}t$ is a zero of the function $\zeta(s)$ in the critical strip. Then removing all but one term from the sum on the right hand side of (38), we have

$$\mathfrak{Re}\left(-\frac{\zeta'(\sigma+\mathrm{i}t)}{\zeta(\sigma+\mathrm{i}t)}\right) < A_3\log t - \mathfrak{Re}\frac{1}{(\sigma+\mathrm{i}t)-(\beta+\mathrm{i}t)} = A_3\log t - \frac{1}{\sigma-\beta} \quad \text{if } 1<\sigma\le 2. \tag{40}$$

Combining (36), (37), (39) and (40), we obtain

$$0 < \frac{3}{\sigma-1} + 3A_1 + (4A_3+A_4)\log t - \frac{4}{\sigma-\beta} \quad \text{if } 1<\sigma\le 2.$$

In other words, there exists a positive absolute constant A_5 such that

$$\frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + A_5\log t \quad \text{if } 1<\sigma\le 2.$$

Let $\sigma = 1 + \delta/\log t$, where $\delta > 0$ will be chosen later, sufficiently small to guarantee that $1 < \sigma \leq 2$. Then elementary calculation gives the inequality

$$\beta < 1 - \frac{\delta(1-A_5\delta)}{(3+A_5\delta)\log t}.$$

We now choose δ in terms of A_5 to conclude that there exists a positive absolute constant c such that

$$\beta < 1 - \frac{c}{\log t}. \tag{41}$$

In conclusion, we have shown that if $t \geq 2$ and $\beta + \mathrm{i}t$ is a zero of the function $\zeta(s)$ in the critical strip, then the inequality (41) must hold. Theorem 6F follows immediately.

6.7. Counting Zeros in the Critical Strip

The starting point of our discussion is based on the Argument principle. Suppose that the function $F(s)$ is analytic, apart from a finite number of poles, in the closure of a domain D bounded by a simple closed positively oriented Jordan curve C. Suppose further that $F(s)$ has no zeros or poles on C. Then

$$\frac{1}{2\pi\mathrm{i}}\int_C \frac{F'(s)}{F(s)}\,\mathrm{d}s = \frac{1}{2\pi}\Delta_C \arg F(s)$$

represents the total number of zeros of $F(s)$ in D minus the total number of poles of $F(s)$ in D, counted with multiplicities. Here $\Delta_C \arg F(s)$ denotes the change of argument of the function $F(s)$ along C.

It is convenient to use the function $\xi(s)$, since it is entire and its zeros are precisely the zeros of $\zeta(s)$ in the critical strip. To calculate $N(T)$, it is convenient to take the domain $(-1,2)\times(0,T)$, so that C is the rectangular path passing through the vertices

$$2, \quad 2+\mathrm{i}T, \quad -1+\mathrm{i}T, \quad -1$$

in the anticlockwise direction. If no zeros of $\zeta(s)$ has imaginary part T, then

$$N(T) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s = \frac{1}{2\pi}\Delta_C \arg \xi(s).$$

Let us now divide C into the following parts. First, let L_1 denote the line segment from -1 to 2. Next, let L_2 denote the line segment from 2 to $2+\mathrm{i}T$, followed by the line segment from $2+\mathrm{i}T$ to $\frac{1}{2}+\mathrm{i}T$. Finally, let L_3 denote the line segment from $\frac{1}{2}+\mathrm{i}T$ to $-1+\mathrm{i}T$, followed by the line segment from $-1+\mathrm{i}T$ to -1.

Since $\xi(s)$ is real and non-zero on L_1, clearly $\Delta_{L_1}\arg\xi(s)=0$. On the other hand,

$$\xi(\sigma+\mathrm{i}t)=\xi(1-\sigma-\mathrm{i}t)=\overline{\xi(1-\sigma+\mathrm{i}t)},$$

so that $\Delta_{L_2}\arg\xi(s)=\Delta_{L_3}\arg\xi(s)$. If we write $L=L_2$, so that L denotes the line segment from 2 to $2+\mathrm{i}T$, followed by the line segment from $2+\mathrm{i}T$ to $\frac{1}{2}+\mathrm{i}T$, then

$$\pi N(T)=\Delta_L\arg\xi(s). \tag{42}$$

Recall that

$$\xi(s)=(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\zeta(s).$$

It follows that

$$\Delta_L\arg\xi(s)=\Delta_L\arg(s-1)+\Delta_L\arg\pi^{-s/2}+\Delta_L\arg\Gamma\left(\frac{s}{2}+1\right)+\Delta_L\arg\zeta(s). \tag{43}$$

Clearly

$$\Delta_L\arg(s-1)=\arg\left(-\frac{1}{2}+\mathrm{i}T\right)=\frac{1}{2}\pi+O(T^{-1}) \tag{44}$$

and

$$\Delta_L\arg\pi^{-s/2}=\Delta_L\left(-\frac{1}{2}t\log\pi\right)=-\frac{1}{2}T\log\pi. \tag{45}$$

On the other hand,

$$\Delta_L\arg\Gamma\left(\frac{s}{2}+1\right)=\mathfrak{Im}\log\Gamma\left(\frac{5}{4}+\frac{1}{2}\mathrm{i}T\right).$$

By Stirling's formula,

$$\log\Gamma\left(\frac{5}{4}+\frac{1}{2}\mathrm{i}T\right)=\left(\frac{3}{4}+\frac{1}{2}\mathrm{i}T\right)\log\left(\frac{5}{4}+\frac{1}{2}\mathrm{i}T\right)-\frac{5}{4}-\frac{1}{2}\mathrm{i}T+\frac{1}{2}\log\pi+O(T^{-1}),$$

so that

$$\Delta_L\arg\Gamma\left(\frac{s}{2}+1\right)=\frac{T}{2}\log\frac{T}{2}+\frac{3}{8}\pi-\frac{T}{2}+O(T^{-1}). \tag{46}$$

Combining (42)–(46), we have

$$\begin{aligned}N(T)&=\frac{1}{2}-\frac{T}{2\pi}\log\pi+\frac{T}{2\pi}\log\frac{T}{2}+\frac{3}{8}-\frac{T}{2\pi}+S(T)+O(T^{-1})\\&=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+S(T)+O(T^{-1}),\end{aligned}$$

where

$$\pi S(T)=\Delta_L\arg\zeta(s).$$

To prove Theorem 6D, it suffices to prove the following result.

THEOREM 6P. *We have* $S(T)=O(\log T)$ *as* $T\to\infty$.

Note first of all that $\arg\zeta(2) = 0$. On the other hand,

$$\arg\zeta(s) = \tan^{-1}\left(\frac{\mathfrak{Im}\zeta(s)}{\mathfrak{Re}\zeta(s)}\right)$$

and $\mathfrak{Re}\zeta(s) \neq 0$ on the line $\sigma = 2$. It follows that

$$|\arg\zeta(2+\mathrm{i}T)| < \frac{\pi}{2},$$

and so $\Delta_{[2,2+\mathrm{i}T]}\arg\zeta(s) = O(1)$. Hence we may assume, without loss of generality, that L is the line segment from $2+\mathrm{i}T$ to $\frac{1}{2}+\mathrm{i}T$. We shall give two proofs, the first of which uses Jensen's formula.

Suppose that $\mathfrak{Re}\zeta(s)$ vanishes q times on the line segment from $2+\mathrm{i}T$ to $\frac{1}{2}+\mathrm{i}T$. Then this line segment can be divided into $q+1$ parts, where in each subinterval, $\mathfrak{Re}\zeta(s)$ may vanish only at one or both of the endpoints and has constant sign strictly in between, so that the variation of $\arg\zeta(s)$ in each such subinterval does not exceed π. It follows that

$$S(T) \ll (q+1)\pi + \frac{1}{2}\pi. \tag{47}$$

To prove our result, it remains to find a suitable bound for q. For $s = \sigma + \mathrm{i}T$, we have

$$\mathfrak{Re}\zeta(s) = \frac{1}{2}(\zeta(\sigma+\mathrm{i}T) + \zeta(\sigma-\mathrm{i}T)).$$

Let T be fixed, and consider the function

$$f_T(s) = \frac{1}{2}(\zeta(s+\mathrm{i}T) + \zeta(s-\mathrm{i}T))$$

(note that we no longer insist that $s = \sigma + \mathrm{i}T$). Then q is the number of zeros of $f_T(s)$ on the line segment from $1/2$ to 2, and so is bounded above by the number of zeros of $f_T(s)$ in the disc $|s-2| \leq 3/2$. In other words,

$$q \leq n\left(\frac{3}{2}\right), \tag{48}$$

where, for every $r \geq 0$, $n(r)$ denotes the number of zeros of $f_T(s)$ in the disc $|s-2| \leq r$. By Jensen's formula and noting that we may assume that $\zeta(\frac{1}{2}+\mathrm{i}T) \neq 0$, we have

$$\int_0^{7/4} \frac{n(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f_T\left(2+\frac{7}{4}\mathrm{e}^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}\theta - \log|f_T(2)|. \tag{49}$$

On the other hand,

$$\int_0^{7/4} \frac{n(r)}{r}\,\mathrm{d}r \geq \int_{3/2}^{7/4} \frac{n(r)}{r}\,\mathrm{d}r \geq n\left(\frac{3}{2}\right)\int_{3/2}^{7/4} \frac{1}{r}\,\mathrm{d}r = n\left(\frac{3}{2}\right)\log\frac{7}{6}. \tag{50}$$

Observe that

$$\begin{aligned} |f_T(2)| &= \left|\frac{1}{2}(\zeta(2+\mathrm{i}T) + \zeta(2-\mathrm{i}T))\right| = |\mathfrak{Re}\zeta(2+\mathrm{i}T)| \\ &= \left|\mathfrak{Re}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2+\mathrm{i}T}}\right)\right| \geq 1 - \sum_{n=2}^{\infty}\frac{1}{n^2} = 2 - \frac{\pi^2}{6} > 0, \end{aligned}$$

so that

$$-\log|f_T(2)| = O(1). \tag{51}$$

Finally, recall that $|\zeta(s)| \ll T^{3/4}$ for every $\sigma \geq 1/4$. It follows that for every $\theta \in [0, 2\pi]$, we have

$$\left| f_T\left(2 + \frac{7}{4}\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leq \frac{1}{2}\left(\left|\zeta\left(2 + \frac{7}{4}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}T\right)\right| + \left|\zeta\left(2 + \frac{7}{4}\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}T\right)\right|\right) \ll T^{3/4},$$

so that

$$\log\left| f_T\left(2 + \frac{7}{4}\mathrm{e}^{\mathrm{i}\theta}\right)\right| \ll \log T. \tag{52}$$

Combining (49)–(52), we conclude that

$$n\left(\frac{3}{2}\right) \ll \log T. \tag{53}$$

Theorem 6P now follows on combining (47), (48) and (53).

The starting point of our second proof of Theorem 6P is the observation that

$$\Delta_{[\frac{1}{2}+\mathrm{i}T, 2+\mathrm{i}T]} \arg\zeta(s) = \int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} \mathfrak{Im} \frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s,$$

since $\arg\zeta(s) = \mathfrak{Im}\log\zeta(s)$. We therefore need to study the logarithmic derivative of $\zeta(s)$ on the line segment between $\frac{1}{2} + \mathrm{i}T$ and $2 + \mathrm{i}T$, and show that

$$\int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} \mathfrak{Im} \frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s = O(\log T).$$

This approach has the added bonus of providing some intermediate results which are useful in the deduction of the asymptotic formula given in Theorem 6E. Recall the inequality (38). If we write $s = 2 + \mathrm{i}T$, where $T \geq 2$, then

$$\mathfrak{Re}\left(-\frac{\zeta'(2+\mathrm{i}T)}{\zeta(2+\mathrm{i}T)}\right) < A_3 \log T - \sum_\rho \mathfrak{Re}\left(\frac{1}{\rho} + \frac{1}{2+\mathrm{i}T-\rho}\right),$$

where A_3 is a positive absolute constant. Clearly

$$\mathfrak{Re}\left(-\frac{\zeta'(2+\mathrm{i}T)}{\zeta(2+\mathrm{i}T)}\right) = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^2}\cos(T\log n) = O\left(\sum_{n=1}^\infty \frac{\log n}{n^2}\right) = O(1),$$

with the immediate consequence that

$$\sum_\rho \mathfrak{Re}\left(\frac{1}{\rho} + \frac{1}{2+\mathrm{i}T-\rho}\right) = O(\log T).$$

Writing $\rho = \beta + \mathrm{i}\gamma$, where $\beta, \gamma \in \mathbb{R}$, we see that

$$\mathfrak{Re}\frac{1}{\rho} > 0 \qquad \text{and} \qquad \mathfrak{Re}\frac{1}{2+\mathrm{i}T-\rho} = \frac{2-\beta}{(2-\beta)^2 + (T-\gamma)^2} \geq \frac{1}{4 + (T-\gamma)^2}.$$

We have proved the following result.

THEOREM 6Q. *For all sufficiently large positive real numbers T, we have*

$$\sum_\rho \frac{1}{1 + (T-\gamma)^2} = O(\log T),$$

where ρ denotes the zeros of the function $\zeta(s)$ in the critical strip.

This has two immediate consequences. Their proofs are left as exercises.

THEOREM 6R. *For all sufficiently large positive real numbers T, the number of zeros of the function $\zeta(s)$ in the critical strip with $|\gamma - T| < 1$ is $O(\log T)$.*

THEOREM 6S. *For all sufficiently large positive real numbers T, we have*

$$\sum_{\substack{\rho \\ |\gamma-T|\ge 1}} \frac{1}{(T-\gamma)^2} = O(\log T),$$

where ρ denotes the zeros of the function $\zeta(s)$ in the critical strip.

The crucial estimate is given by the following result. The range for σ is greater than for our present need, but will be necessary for later use.

THEOREM 6T. *For every $s = \sigma + \mathrm{i}T$, where $-1 \le \sigma \le 2$ and where T is sufficiently large and $T \ne \gamma$ for any zero $\rho = \beta + \mathrm{i}\gamma$ of the function $\zeta(s)$ in the critical strip, we have*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\substack{\rho \\ |\gamma-T|<1}} \frac{1}{s-\rho} + O(\log T), \tag{54}$$

where ρ denotes the zeros of the function $\zeta(s)$ in the critical strip.

Proof. We start with the formula

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{s-\rho}\right), \tag{55}$$

given in Theorem 6N. Writing $s = 2 + \mathrm{i}T$, we have

$$\frac{\zeta'(2+\mathrm{i}T)}{\zeta(2+\mathrm{i}T)} = B - \frac{1}{1+\mathrm{i}T} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(2+\frac{1}{2}\mathrm{i}T)}{\Gamma(2+\frac{1}{2}\mathrm{i}T)} + \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{2+\mathrm{i}T-\rho}\right). \tag{56}$$

Note now that $\zeta'(2+\mathrm{i}T)/\zeta(2+\mathrm{i}T) = O(1)$ and $\Gamma'(2+\frac{1}{2}\mathrm{i}T)/\Gamma(2+\frac{1}{2}\mathrm{i}T) = O(\log T)$ for every T under consideration, and $\Gamma'(\frac{s}{2}+1)/\Gamma(\frac{s}{2}+1) = O(\log T)$ for every s under consideration. It follows on subtracting (56) from (55) that

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}T-\rho}\right) + O(\log T) \\ &= \sum_{\substack{\rho \\ |\gamma-T|<1}} \frac{1}{s-\rho} - \sum_{\substack{\rho \\ |\gamma-T|<1}} \frac{1}{2+\mathrm{i}T-\rho} + \sum_{\substack{\rho \\ |\gamma-T|\ge 1}} \left(\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}T-\rho}\right) + O(\log T). \end{aligned} \tag{57}$$

On the one hand, every zero ρ of the function $\zeta(s)$ in the critical strip satisfies $|2+\mathrm{i}T-\rho| > 1$. Hence

$$\sum_{\substack{\rho \\ |\gamma-T|<1}} \frac{1}{2+\mathrm{i}T-\rho} = O\left(\sum_{\substack{\rho \\ |\gamma-T|<1}} 1\right) = O(\log T), \tag{58}$$

in view of Theorem 6R. On the other hand, if $|\gamma - T| \ge 1$, then

$$\left|\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}T-\rho}\right| = \frac{2-\sigma}{|(s-\rho)(2+\mathrm{i}T-\rho)|} \le \frac{3}{(T-\gamma)^2},$$

and so

$$\sum_{\substack{\rho \\ |\gamma-T|\geq 1}} \left(\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}T-\rho} \right) = O\left(\sum_{\substack{\rho \\ |\gamma-T|\geq 1}} \frac{1}{(T-\gamma)^2} \right) = O(\log T), \tag{59}$$

in view of Theorem 6S. The inequality (54) now follows on combining (57)–(59). ○

Taking imaginary parts in the inequality (54) on the line segment between $\frac{1}{2}+\mathrm{i}T$ and $2+\mathrm{i}T$, where T is sufficiently large and $T \neq \gamma$ for any zero $\rho = \beta + \mathrm{i}\gamma$ of the function $\zeta(s)$ in the critical strip, gives

$$\mathfrak{Im}\frac{\zeta'(s)}{\zeta(s)} = \sum_{\substack{\rho \\ |\gamma-T|<1}} \mathfrak{Im}\frac{1}{s-\rho} + O(\log T).$$

Combining this with the simple observation that

$$\left| \int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} \mathfrak{Im}\frac{1}{s-\rho}\,\mathrm{d}s \right| = \left| \Delta_{[\frac{1}{2}+\mathrm{i}T,2+\mathrm{i}T]} \arg(s-\rho) \right| < \pi$$

and Theorem 6R, we obtain

$$\int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} \mathfrak{Im}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s = O\left(\sum_{\substack{\rho \\ |\gamma-T|<1}} 1 \right) + O(\log T) = O(\log T).$$

This completes the proof of Theorem 6P.

6.8. An Asymptotic Formula

In this section, we shall establish the asymptotic formula in Theorem 6E when $X > \mathrm{e}$. Here the starting point is the simple observation that

$$\psi_0(X) = {\sum_{n\leq X}}' \Lambda(n),$$

where $\sum'$ denotes that the term in the sum corresponding to $n = X$ is $\frac{1}{2}\Lambda(n)$. Analogous to the discussion for $\psi_1(X)$ in Chapter 5, we want to write

$$\psi_0(X) = {\sum_{n\leq X}}' \Lambda(n) = \sum_{n=1}^{\infty} \Lambda(n) I\left(\frac{X}{n}\right), \tag{60}$$

where the function

$$I(Y) = \begin{cases} 0 & \text{if } 0 < Y < 1, \\ 1/2 & \text{if } Y = 1, \\ 1 & \text{if } Y > 1, \end{cases} \tag{61}$$

brings the function $\zeta(s)$ into play.

The following is a suitable analogue of Theorem 5F. However, the proof is much more complicated since we do not have absolute convergence.

THEOREM 6U. *Suppose that* $Y>0$ *and* $c>0$*. Let*

$$I(Y)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{Y^s}{s}\,\mathrm{d}s,$$

where the integral in the case $Y=1$ *is defined to be the limit of*

$$I(Y,T)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\frac{Y^s}{s}\,\mathrm{d}s$$

as $T\to\infty$*. Then* (60) *and* (61) *hold. Furthermore, for every* $T>0$*, we have*

$$|I(Y)-I(Y,T)|\le\begin{cases}Y^c\min\{1,(\pi T|\log Y|)^{-1}\} & \text{if } Y\ne1,\\ c(\pi T)^{-1} & \text{if } Y=1.\end{cases}\tag{62}$$

Proof. For every $T_1,T_2>0$, write

$$I(Y,T_1,T_2)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T_1}^{c+\mathrm{i}T_2}\frac{Y^s}{s}\,\mathrm{d}s.$$

We shall consider three cases, corresponding to $Y>1$, $0<Y<1$ and $Y=1$.

Suppose first of all that $Y>1$. We consider the rectangular path with vertices

$$c-\mathrm{i}T_1,\quad c+\mathrm{i}T_2,\quad -u+\mathrm{i}T_2,\quad -u-\mathrm{i}T_1,$$

where $u>0$, followed in the anticlockwise direction. Applying Cauchy's residue theorem, we obtain

$$I(Y,T_1,T_2)-1=\frac{1}{2\pi\mathrm{i}}\int_{-u-\mathrm{i}T_1}^{-u+\mathrm{i}T_2}\frac{Y^s}{s}\,\mathrm{d}s+\frac{1}{2\pi\mathrm{i}}\int_{-u+\mathrm{i}T_2}^{c+\mathrm{i}T_2}\frac{Y^s}{s}\,\mathrm{d}s-\frac{1}{2\pi\mathrm{i}}\int_{-u-\mathrm{i}T_1}^{c-\mathrm{i}T_1}\frac{Y^s}{s}\,\mathrm{d}s.\tag{63}$$

On the vertical edge $\sigma=-u$, we have $|Y^s/s|\le Y^{-u}/u$, and so

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{-u-\mathrm{i}T_1}^{-u+\mathrm{i}T_2}\frac{Y^s}{s}\,\mathrm{d}s\right|\le\frac{(T_1+T_2)Y^{-u}}{2\pi u}.\tag{64}$$

On the horizontal edge $t=T_2$, we have $|Y^s/s|\le Y^\sigma/T_2$, and so

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{-u+\mathrm{i}T_2}^{c+\mathrm{i}T_2}\frac{Y^s}{s}\,\mathrm{d}s\right|\le\frac{1}{2\pi}\int_{-\infty}^{c}\frac{Y^\sigma}{T_2}\,\mathrm{d}\sigma=\frac{Y^c}{2\pi T_2\log Y}.\tag{65}$$

On the horizontal edge $t=-T_1$, we have $|Y^s/s|\le Y^\sigma/T_1$, and so

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{-u-\mathrm{i}T_1}^{c-\mathrm{i}T_1}\frac{Y^s}{s}\,\mathrm{d}s\right|\le\frac{1}{2\pi}\int_{-\infty}^{c}\frac{Y^\sigma}{T_1}\,\mathrm{d}\sigma=\frac{Y^c}{2\pi T_1\log Y}.\tag{66}$$

Combining (63)–(66), we obtain

$$|I(Y,T_1,T_2)-1|\le\frac{(T_1+T_2)Y^{-u}}{2\pi u}+\frac{Y^c}{2\pi T_1\log Y}+\frac{Y^c}{2\pi T_2\log Y}.$$

Since the left hand side is independent of u, it follows on letting $u\to\infty$ that

$$|I(Y,T_1,T_2)-1|\le\frac{Y^c}{2\pi T_1\log Y}+\frac{Y^c}{2\pi T_2\log Y}.$$

Letting $T_1, T_2 \to \infty$ gives (61). Letting $T = T_1 = T_2$ gives one of the inequalities in (62). To deduce the other inequality, we use the circular arc $A^-(c,T)$ centred at $s = 0$ and passing from $c - \mathrm{i}T$ to $c + \mathrm{i}T$ on the left of the line $\sigma = c$, as in the proof of Theorem 5F. Then Cauchy's residue theorem gives

$$I(Y,T) - I(Y) = I(Y,T) - 1 = \frac{1}{2\pi\mathrm{i}} \int_{A^-(c,T)} \frac{Y^s}{s} \,\mathrm{d}s. \tag{67}$$

On the circular arc $A^-(c,T)$, we have $|Y^s/s| \le Y^c/R$, where $R = (c^2 + T^2)^{1/2}$ is the radius of $A^-(c,T)$. It follows that

$$\left| \frac{1}{2\pi\mathrm{i}} \int_{A^-(c,T)} \frac{Y^s}{s} \,\mathrm{d}s \right| \le \frac{1}{2\pi} \frac{Y^c}{R} 2\pi R = Y^c. \tag{68}$$

The inequality $|I(Y) - I(Y,T)| \le Y^c$ now follows on combining (67) and (68).

Suppose next that $0 < Y < 1$. We consider the rectangular path with vertices

$$c - \mathrm{i}T_1, \quad c + \mathrm{i}T_2, \quad u + \mathrm{i}T_2, \quad u - \mathrm{i}T_1,$$

where $u > 0$, followed in the clockwise direction. Applying Cauchy's integral theorem, we obtain

$$I(Y,T_1,T_2) = \frac{1}{2\pi\mathrm{i}} \int_{u-\mathrm{i}T_1}^{u+\mathrm{i}T_2} \frac{Y^s}{s} \,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T_1}^{u-\mathrm{i}T_1} \frac{Y^s}{s} \,\mathrm{d}s - \frac{1}{2\pi\mathrm{i}} \int_{c+\mathrm{i}T_2}^{u+\mathrm{i}T_2} \frac{Y^s}{s} \,\mathrm{d}s. \tag{69}$$

On the vertical edge $\sigma = u$, we have $|Y^s/s| \le Y^u/u$, and so

$$\left| \frac{1}{2\pi\mathrm{i}} \int_{u-\mathrm{i}T_1}^{u+\mathrm{i}T_2} \frac{Y^s}{s} \,\mathrm{d}s \right| \le \frac{(T_1 + T_2)Y^u}{2\pi u}. \tag{70}$$

On the horizontal edge $t = -T_1$, we have $|Y^s/s| \le Y^\sigma/T_1$, and so

$$\left| \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T_1}^{u-\mathrm{i}T_1} \frac{Y^s}{s} \,\mathrm{d}s \right| \le \frac{1}{2\pi} \int_c^\infty \frac{Y^\sigma}{T_1} \,\mathrm{d}\sigma = -\frac{Y^c}{2\pi T_1 \log Y} = \frac{Y^c}{2\pi T_1 |\log Y|}. \tag{71}$$

On the horizontal edge $t = T_2$, we have $|Y^s/s| \le Y^\sigma/T_2$, and so

$$\left| \frac{1}{2\pi\mathrm{i}} \int_{c+\mathrm{i}T_2}^{u+\mathrm{i}T_2} \frac{Y^s}{s} \,\mathrm{d}s \right| \le \frac{1}{2\pi} \int_c^\infty \frac{Y^\sigma}{T_2} \,\mathrm{d}\sigma = -\frac{Y^c}{2\pi T_2 \log Y} = \frac{Y^c}{2\pi T_2 |\log Y|}. \tag{72}$$

Combining (69)–(72), we obtain

$$|I(Y,T_1,T_2)| \le \frac{(T_1 + T_2)Y^u}{2\pi u} + \frac{Y^c}{2\pi T_1 |\log Y|} + \frac{Y^c}{2\pi T_2 |\log Y|}.$$

Since the left hand side is independent of u, it follows on letting $u \to \infty$ that

$$|I(Y,T_1,T_2)| \le \frac{Y^c}{2\pi T_1 |\log Y|} + \frac{Y^c}{2\pi T_2 |\log Y|}.$$

Letting $T_1, T_2 \to \infty$ gives (61). Letting $T = T_1 = T_2$ gives one of the inequalities in (62). To deduce the other inequality, we use the circular arc $A^+(c,T)$ centred at $s = 0$ and passing from $c - \mathrm{i}T$ to $c + \mathrm{i}T$ on the right of the line $\sigma = c$, as in the proof of Theorem 5F. Then Cauchy's integral theorem gives

$$I(Y,T) - I(Y) = I(Y,T) = \frac{1}{2\pi\mathrm{i}} \int_{A^+(c,T)} \frac{Y^s}{s} \,\mathrm{d}s. \tag{73}$$

On the circular arc $A^+(c,T)$, we have $|Y^s/s| \le Y^c/R$, where $R=(c^2+T^2)^{1/2}$ is the radius of $A^+(c,T)$. It follows that

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{A^+(c,T)}\frac{Y^s}{s}\,\mathrm{d}s\right| \le \frac{1}{2\pi}\frac{Y^c}{R}2\pi R = Y^c. \tag{74}$$

The inequality $|I(Y)-I(Y,T)| \le Y^c$ now follows on combining (73) and (74).

Suppose finally that $Y=1$. Then

$$I(1,T)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\frac{\mathrm{d}s}{s}=\frac{1}{2\pi}\int_{-T}^{T}\frac{\mathrm{d}t}{c+\mathrm{i}t}=\frac{1}{2\pi}\int_{-T}^{T}\frac{c-\mathrm{i}t}{c^2+t^2}\,\mathrm{d}t.$$

Note that the imaginary part of the integrand of the last integral is an odd function, while the real part is an even function. It follows that

$$I(1,T)=\frac{1}{\pi}\int_0^T\frac{c}{c^2+t^2}\,\mathrm{d}t \to \frac{1}{2} \quad \text{as } T\to\infty.$$

On the other hand, we have

$$|I(1)-I(1,T)|=\left|\frac{1}{2}-I(1,T)\right|=\frac{1}{\pi}\int_T^\infty\frac{c}{c^2+t^2}\,\mathrm{d}t \le \frac{1}{\pi}\int_T^\infty\frac{c}{t^2}\,\mathrm{d}t = c(\pi T)^{-1}.$$

This completes the proof. ○

In view of the identity (60), it is now reasonable to compare $\psi_0(X)$ with the sum

$$\begin{aligned}\psi(X,T)&=\sum_{n=1}^\infty\Lambda(n)I\left(\frac{X}{n},T\right)=\frac{1}{2\pi\mathrm{i}}\sum_{n=1}^\infty\Lambda(n)\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\frac{X^s}{sn^s}\,\mathrm{d}s\\&=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\left(\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s}\right)\frac{X^s}{s}\,\mathrm{d}s=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s.\end{aligned}$$

Using Theorem 6U, we see that

$$\begin{aligned}|\psi_0(X)-\psi(X,T)|&\le\sum_{n=1}^\infty\Lambda(n)\left|I\left(\frac{X}{n}\right)-I\left(\frac{X}{n},T\right)\right|\\&\le\sum_{\substack{n=1\\n\ne X}}^\infty\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\}+c(\pi T)^{-1}\Lambda(X),\end{aligned} \tag{75}$$

with the understanding that the last term is present only if X is a prime power.

Let $X>\mathrm{e}$ be given and fixed. We shall choose

$$c=1+(\log X)^{-1}, \qquad \text{so that} \qquad X^c=\mathrm{e}X. \tag{76}$$

Note that $c<2$. We can write

$$\sum_{\substack{n=1\\n\ne X}}^\infty\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\}=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4, \tag{77}$$

where

$$\sum\nolimits_1=\sum_{n\le 3X/4}\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\},$$
$$\sum\nolimits_2=\sum_{3X/4<n<X}\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\},$$
$$\sum\nolimits_3=\sum_{X<n<4X/3}\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\},$$
$$\sum\nolimits_4=\sum_{n\ge 4X/3}\Lambda(n)\left(\frac{X}{n}\right)^c\min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\}.$$

Suppose first of all that $n\le 3X/4$ or $n\ge 4X/3$. Then it is easy to see that $|\log(X/n)|\ge\log(4/3)$. In view of (76) and (37), we have

$$\sum\nolimits_1+\sum\nolimits_4\ll XT^{-1}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^c}=XT^{-1}\left(-\frac{\zeta'(c)}{\zeta(c)}\right)\ll XT^{-1}\log X. \tag{78}$$

Suppose next that $3X/4<n<X$. In this case, let X_1 denote the largest prime power less than X. We may assume, without loss of generality, that $3X/4<X_1<X$, for otherwise we have $\sum_2=0$. For the term $n=X_1$, we have

$$\log\frac{X}{n}=-\log\frac{X_1}{X}=-\log\left(1-\frac{X-X_1}{X}\right)\ge\frac{X-X_1}{X}.$$

It follows that the contribution of this term to the sum $\sum_2$ is

$$\ll\Lambda(X_1)\min\left\{1,\frac{X}{T(X-X_1)}\right\}\ll(\log X)\min\left\{1,\frac{X}{T(X-X_1)}\right\}.$$

The other terms form a subcollection of $n=X_1-m$, where $0<m<X/4$, and we have

$$\log\frac{X}{n}\ge\log\frac{X_1}{n}=-\log\frac{n}{X_1}=-\log\left(1-\frac{m}{X_1}\right)\ge\frac{m}{X_1}.$$

It follows that the contribution of these terms to the sum $\sum_2$ is

$$\ll\sum_{0<m<X/4}\Lambda(X_1-m)\frac{X_1}{Tm}\ll XT^{-1}\sum_{0<m<X/4}\frac{\Lambda(X_1-m)}{m}\ll XT^{-1}(\log X)^2.$$

Hence

$$\sum\nolimits_2\ll XT^{-1}(\log X)^2+(\log X)\min\left\{1,\frac{X}{T\langle X\rangle}\right\}, \tag{79}$$

where $\langle X\rangle$ denotes the distance of X to the nearest prime power.

Suppose finally that $X<n<4X/3$. In this case, let X_2 denote the smallest prime power greater than X. We may assume, without loss of generality, that $X<X_2<4X/3$, for otherwise we have $\sum_3=0$. For the term $n=X_2$, we have

$$\left|\log\frac{X}{n}\right|=\log\frac{X_2}{X}=\log\left(1+\frac{X_2-X}{X}\right)\ge\frac{X_2-X}{X}.$$

It follows that the contribution of this term to the sum $\sum_3$ is

$$\ll \Lambda(X_2)\min\left\{1,\frac{X}{T(X_2-X)}\right\} \ll (\log X)\min\left\{1,\frac{X}{T(X_2-X)}\right\}.$$

The other terms form a subcollection of $n = X_2 + m$, where $0 < m < X/3$, and we have

$$\left|\log\frac{X}{n}\right| \geq \left|\log\frac{X_2}{n}\right| = \log\frac{n}{X_2} = \log\left(1+\frac{m}{X_2}\right) \geq \frac{m}{X_2}.$$

It follows that the contribution of these terms to the sum $\sum_3$ is

$$\ll \sum_{0<m<X/3} \Lambda(X_2+m)\frac{X_2}{Tm} \ll XT^{-1}\sum_{0<m<X/3}\frac{\Lambda(X_2+m)}{m} \ll XT^{-1}(\log X)^2.$$

Hence

$$\sum\nolimits_3 \ll XT^{-1}(\log X)^2 + (\log X)\min\left\{1,\frac{X}{T\langle X\rangle}\right\}. \tag{80}$$

Combining (77)–(80), we conclude that

$$\sum_{\substack{n=1\\ n\neq X}}^{\infty} \Lambda(n)\left(\frac{X}{n}\right)^c \min\left\{1,\left(\pi T\left|\log\frac{X}{n}\right|\right)^{-1}\right\} \ll XT^{-1}(\log X)^2 + (\log X)\min\left\{1,\frac{X}{T\langle X\rangle}\right\},$$

and so it follows from (75) and (76) that

$$|\psi_0(X)-\psi(X,T)| \ll XT^{-1}(\log X)^2 + (\log X)\min\left\{1,\frac{X}{T\langle X\rangle}\right\} \quad \text{if } c = 1+(\log X)^{-1}. \tag{81}$$

We now need to study the term $\psi(X,T)$.

Consider a rectangular path with vertices

$$c-\mathrm{i}T,\quad c+\mathrm{i}T,\quad -U+\mathrm{i}T,\quad -U-\mathrm{i}T,$$

followed in the anticlockwise direction and where U and T are chosen carefully to satisfy the following two conditions:

(i) U is a large odd positive integer to ensure that the left edge of the rectangular path passes halfway between two consecutive trivial zeros of $\zeta(s)$.

(ii) T is chosen so that $|\gamma - T| \gg (\log T)^{-1}$ for any zero $\rho = \beta \pm \mathrm{i}\gamma$ of $\zeta(s)$ in the critical strip. This is clearly possible by varying T by a bounded amount, in view of Theorem 6R.

Applying Cauchy's residue theorem, we obtain

$$\begin{aligned}\psi(X,T) = {} & \frac{1}{2\pi\mathrm{i}}\int_{-U+\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}}\int_{-U-\mathrm{i}T}^{-U+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s \\ & - \frac{1}{2\pi\mathrm{i}}\int_{-U-\mathrm{i}T}^{c-\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s + X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho\\ |\gamma|<T}}\frac{X^\rho}{\rho} + \sum_{m=1}^{[U/2]}\frac{X^{-2m}}{2m}.\end{aligned} \tag{82}$$

We study first the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{-U+\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s=\frac{1}{2\pi \mathrm{i}}\int_{-U+\mathrm{i}T}^{-1+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s+\frac{1}{2\pi \mathrm{i}}\int_{-1+\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s. \tag{83}$$

For every $s=\sigma+\mathrm{i}T$, where $-1\le\sigma\le 2$, we have

$$\left|\frac{1}{s-\rho}\right|\le\frac{1}{|T-\gamma|}\ll\log T,$$

in view of condition (ii). This, combined with Theorems 6R and 6T, gives the bound

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right|\ll(\log T)^2.$$

It follows on recalling $X^c=\mathrm{e}X$ and $c<2$ that

$$\frac{1}{2\pi \mathrm{i}}\int_{-1+\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s\ll\frac{(\log T)^2}{T}\int_{-\infty}^{c}X^\sigma\,\mathrm{d}\sigma\ll\frac{X(\log T)^2}{T\log X}. \tag{84}$$

To study the first integral on the right hand side of (83), we need the following estimate.

THEOREM 6V. *Suppose that* $\mathfrak{Res}\le -1$ *and* $|s+2m|\ge 1$ *for every* $m\in\mathbb{N}$. *Then*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right|\ll\log(2|s|).$$

PROOF. We start with the formula

$$\Gamma\left(\frac{1-s}{2}\right)=\pi^{-1/2}2^s\Gamma(1-s)\Gamma\left(\frac{s}{2}\right)\cos\frac{\pi(1-s)}{2}.$$

Combining this with the functional equation (1), we obtain the functional equation in unsymmetric form

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\cos\frac{\pi(1-s)}{2}.$$

On taking logarithmic derivatives, we have

$$\frac{\zeta'(s)}{\zeta(s)}=C-\frac{\Gamma'(1-s)}{\Gamma(1-s)}-\frac{\zeta'(1-s)}{\zeta(1-s)}+\frac{\pi}{2}\tan\frac{\pi(1-s)}{2},$$

where C is a constant. Note next that if $\mathfrak{Res}\le -1$, then $\mathfrak{Re}(1-s)\ge 2$. Furthermore, if $|s+2m|\ge 1$, then $|(1-s)-(2m+1)|\ge 1$, so that $\tan\frac{1}{2}\pi(1-s)$ is bounded. The result now follows on noting that

$$\left|\frac{\Gamma'(1-s)}{\Gamma(1-s)}\right|=O(\log|1-s|)=O(\log 2|s|)\qquad\text{and}\qquad\left|\frac{\zeta'(1-s)}{\zeta(1-s)}\right|=O(1)$$

for $\mathfrak{Re}(1-s)\ge 2$. ○

It now follows from Theorem 6V that

$$\frac{1}{2\pi \mathrm{i}}\int_{-U+\mathrm{i}T}^{-1+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s\ll\frac{\log 2T}{T}\int_{-U}^{-1}X^\sigma\,\mathrm{d}\sigma\ll\frac{\log T}{TX\log X}. \tag{85}$$

Combining (83)–(85), we obtain

$$\frac{1}{2\pi \mathrm{i}}\int_{-U+\mathrm{i}T}^{c+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s \ll \frac{X(\log T)^2}{T\log X}. \tag{86}$$

A similar consideration gives the analogous estimate

$$\frac{1}{2\pi \mathrm{i}}\int_{-U-\mathrm{i}T}^{c-\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s \ll \frac{X(\log T)^2}{T\log X}. \tag{87}$$

Note also that Theorem 6V also leads to the estimate

$$\frac{1}{2\pi \mathrm{i}}\int_{-U-\mathrm{i}T}^{-U+\mathrm{i}T}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{X^s}{s}\,\mathrm{d}s \ll \frac{\log 2U}{U}\int_{-T}^{T} X^{-U}\,\mathrm{d}t \ll \frac{T\log U}{UX^U}. \tag{88}$$

Combining (82) and (86)–(88), we obtain

$$\psi(X,T) = X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho\\ |\gamma|<T}} \frac{X^\rho}{\rho} + \sum_{m=1}^{[U/2]} \frac{X^{-2m}}{2m} + O\left(\frac{X(\log T)^2}{T\log X}\right) + O\left(\frac{T\log U}{UX^U}\right),$$

valid for arbitrarily large values of U. Keeping T fixed and letting $U \to \infty$, we deduce that

$$\psi(X,T) = X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho\\ |\gamma|<T}} \frac{X^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{X^{-2m}}{2m} + O\left(\frac{X(\log T)^2}{T\log X}\right).$$

Combining this with (81), we obtain

$$\psi_0(X) = X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho\\ |\gamma|<T}} \frac{X^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{X^{-2m}}{2m} + R(X,T), \tag{89}$$

where the error term $R(X,T)$ satisfies the bound

$$R(X,T) \ll \frac{X(\log XT)^2}{T} + (\log X)\min\left\{1, \frac{X}{T\langle X\rangle}\right\}. \tag{90}$$

However, we have to recognize that (89) and (90) have been established under a restriction on the value of T which has made it necessary for us to vary its value by a bounded amount. This has the effect of changing the number of terms in the sum over ρ in (89) by $O(\log T)$ terms, in view of Theorem 6R, and each such term clearly contributes at most $O(XT^{-1})$. It follows that the error incurred on relaxing the restriction on T is at most $O(XT^{-1}\log T)$ which is easily absorbed in the error estimate (90). Hence (89) and (90) remain valid for all large values of T.

Note now that for fixed X, we have $R(X,T) \to 0$ as $T \to \infty$. It follows that

$$\psi_0(X) = X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{X^\rho}{\rho} + \sum_{m=1}^{\infty} \frac{X^{-2m}}{2m} = X - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{X^\rho}{\rho} - \log\left(1 - \frac{1}{X^2}\right).$$

6.9. The Prime Number Theorem

The estimates (89) and (90) give us another opportunity to establish the Prime number theorem, in the form $\psi_0(X) \sim X$ as $X \to \infty$. Clearly we need a good estimate for the sum

$$\sum_{\substack{\rho \\ |\gamma|<T}} \frac{X^\rho}{\rho}, \tag{91}$$

as well as an appropriate estimate for the error term $R(X,T)$, by choosing suitably large values for the parameter T.

To obtain a good estimate for the sum (91), the idea here is to make use of the zero-free region given by Theorem 6F. A consequence of this zero-free region is that for every zero $\rho = \beta + \mathrm{i}\gamma$ of $\zeta(s)$ in the critical strip with $|\gamma| < T$, where $T > 2$, we have

$$\beta \le 1 - \frac{c}{\log |\gamma|} \le 1 - \frac{c}{\log T},$$

where c is a positive absolute constant. It follows that for any such zero ρ, we have

$$|X^\rho| = X^\beta \le X\mathrm{e}^{-c(\log X)/(\log T)},$$

and so

$$\sum_{\substack{\rho \\ |\gamma|<T}} \frac{X^\rho}{\rho} \ll X\mathrm{e}^{-c(\log X)/(\log T)} \sum_{\substack{\rho \\ 0<\gamma<T}} \frac{1}{\gamma}.$$

Note next that

$$\sum_{\substack{\rho \\ 0<\gamma<T}} \frac{1}{\gamma} = \int_0^T t^{-1}\,\mathrm{d}N(t) = \frac{N(T)}{T} + \int_0^T t^{-2} N(t)\,\mathrm{d}t, \tag{92}$$

where $N(T)$ denotes the number of zeros $\rho = \beta + \mathrm{i}\gamma$ of $\zeta(s)$ in the critical strip with $0 < \gamma < T$. But $N(T) = O(T\log T)$ for large T by Theorem 6D, so the sum (92) is $\ll (\log T)^2$. Hence

$$\sum_{\substack{\rho \\ |\gamma|<T}} \frac{X^\rho}{\rho} \ll X(\log T)^2 \mathrm{e}^{-c(\log X)/(\log T)}. \tag{93}$$

Combining (89), (90) and (93), we now have

$$\psi_0(X) - X \ll \frac{X(\log XT)^2}{T} + (\log X)\min\left\{1, \frac{X}{T\langle X\rangle}\right\} + X(\log T)^2 \mathrm{e}^{-c(\log X)/(\log T)}.$$

We may assume, without loss of generality, that X is an integer, so that $\langle X\rangle \ge 1$. Then

$$\psi_0(X) - X \ll \frac{X(\log XT)^2}{T} + X(\log T)^2 \mathrm{e}^{-c(\log X)/(\log T)}.$$

We now choose T to satisfy $(\log T)^2 = \log X$, so that $T^{-1} = \mathrm{e}^{-(\log X)^{1/2}}$. Then

$$\psi_0(X) - X \ll X(\log X)^2 \mathrm{e}^{-(\log X)^{1/2}} + X(\log X)\mathrm{e}^{-c(\log X)^{-1/2}} \ll X\mathrm{e}^{-c'(\log X)^{-1/2}},$$

where $c' < \min\{1, c\}$ is a positive absolute constant.

PROBLEMS FOR CHAPTER 6

1. Prove Theorem 6R.
2. Prove Theorem 6S.