

ELEMENTARY MATHEMATICS

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Chapter 9

COMPLEX NUMBERS

9.1. Introduction

It is easy to see that the equation $x^2 + 1 = 0$ has no solution $x \in \mathbb{R}$. In order to “solve” this equation, we have to introduce extra numbers into our number system.

Define the number i by writing $i^2 + 1 = 0$. We then extend the collection of all real numbers by adjoining the number i , which is then combined with the real numbers by the operations addition and multiplication in accordance with the rules of arithmetic for real numbers. The numbers $a + bi$, where $a, b \in \mathbb{R}$, of the extended collection are then added and multiplied in accordance with the rules of arithmetic for real numbers, suitably extended, and the restriction $i^2 + 1 = 0$. Note that the number $a + 0i$, where $a \in \mathbb{R}$, behaves like the real number a .

DEFINITION. We denote by \mathbb{C} the collection of all complex numbers; in other words, the collection of all numbers of the form $a + bi$, where $a, b \in \mathbb{R}$.

ARITHMETIC OF COMPLEX NUMBERS. Consider two complex numbers $a + bi$ and $c + di$, where $a, b, c, d \in \mathbb{R}$. We have the addition rule

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and the multiplication rule

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

A simple consequence is the subtraction rule

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

† This chapter was written at Macquarie University in 1999.

For the division rule, suppose that $c + di \neq 0$. Then $c \neq 0$ or $d \neq 0$, so that $c^2 + d^2 \neq 0$. We have

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Alternatively, write

$$\frac{a + bi}{c + di} = x + yi,$$

where $x, y \in \mathbb{R}$. Then

$$a + bi = (c + di)(x + yi) = (cx - dy) + (cy + dx)i.$$

It follows that

$$\begin{aligned} a &= cx - dy, \\ b &= cy + dx. \end{aligned}$$

This system of simultaneous linear equations has the unique solution

$$x = \frac{ac + bd}{c^2 + d^2} \quad \text{and} \quad y = \frac{bc - ad}{c^2 + d^2},$$

so that

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

However, there is absolutely no need to commit either of these two techniques to memory. For the special case $a = 1$ and $b = 0$ gives

$$\frac{1}{c + di} = \frac{c - di}{c^2 + d^2}.$$

This can be obtained by noting that $(c + di)(c - di) = c^2 + d^2$, so that

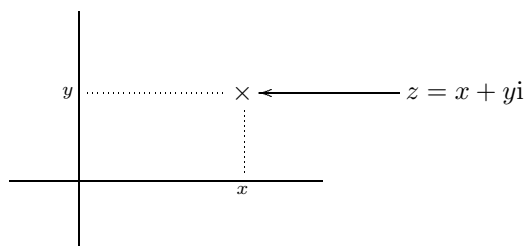
$$\frac{1}{c + di} = \frac{c - di}{(c + di)(c - di)} = \frac{c - di}{c^2 + d^2}.$$

It is also useful to note that i^n has exactly four possible values, with $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.

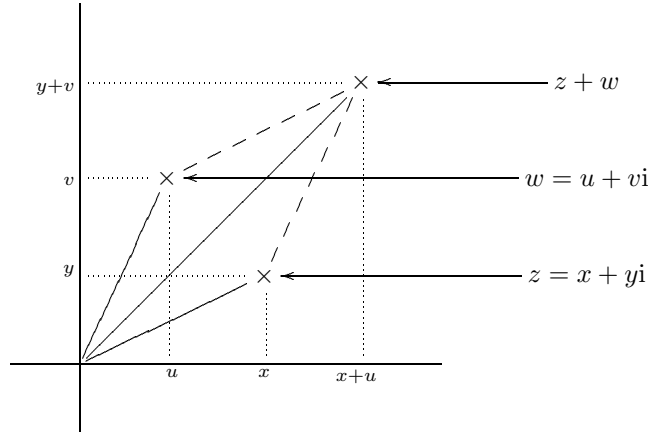
9.2. The Complex Plane

Definition. Suppose that $z = x + yi$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z , and denoted by $x = \Re z$. The real number y is called the imaginary part of z , and denoted by $y = \Im z$.

A useful way of representing complex numbers is to use the Argand diagram. This is made up of the plane together with two axes. The horizontal axis, usually called the real axis, is used to denote the real part of the complex number, while the vertical axis, usually called the imaginary axis, is used to denote the imaginary part of the complex number.



Addition of two complex numbers is then represented by the Argand diagram below.



EXAMPLE 9.2.1. We have $(2 + 3i) + (4 - 5i) = (2 + 4) + (3 - 5)i = 2 - 2i = 2(1 - i)$.

EXAMPLE 9.2.2. We have $(2 + 3i)(4 - 5i) = (2 \times 4 - 3 \times (-5)) + (2 \times (-5) + 3 \times 4)i = 23 + 2i$.

EXAMPLE 9.2.3. We have

$$\frac{2 + 3i}{4 - 5i} = \frac{(2 + 3i)(4 + 5i)}{(4 - 5i)(4 + 5i)} = \frac{-7 + 22i}{41} = -\frac{7}{41} + \frac{22}{41}i.$$

Hence

$$\Re \frac{2 + 3i}{4 - 5i} = -\frac{7}{41} \quad \text{and} \quad \Im \frac{2 + 3i}{4 - 5i} = \frac{22}{41}.$$

EXAMPLE 9.2.4. We have

$$\frac{(1 + 2i)^2}{1 - i} = \frac{-3 + 4i}{1 - i} = \frac{(-3 + 4i)(1 + i)}{(1 - i)(1 + i)} = \frac{-7 + i}{2} = -\frac{7}{2} + \frac{1}{2}i.$$

Hence

$$\Re \frac{(1 + 2i)^2}{1 - i} = -\frac{7}{2} \quad \text{and} \quad \Im \frac{(1 + 2i)^2}{1 - i} = \frac{1}{2}.$$

EXAMPLE 9.2.5. We have $1 + i + i^2 + i^3 = 0$ and $5 + 7i^{2000} = 12$.

Many equations over a complex variable can be studied by similar techniques to equations over a real variable. Here we give two examples to illustrate this point.

EXAMPLE 9.2.6. Suppose that $(1 + 3i)z + 2 + 2i = 3 + i$. Then subtracting $2 + 2i$ from both sides, we obtain $(1 + 3i)z = 1 - i$, and so

$$z = \frac{1 - i}{1 + 3i} = \frac{(1 - i)(1 - 3i)}{(1 + 3i)(1 - 3i)} = \frac{-2 - 4i}{10} = -\frac{1}{5} - \frac{2}{5}i.$$

EXAMPLE 9.2.7. Suppose that $z^2 + 2z + 10 = 0$. Then following the technique for solving quadratic equations, we have

$$z = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i.$$

Alternatively, we observe that

$$z^2 + 2z + 10 = z^2 + 2z + 1 + 9 = (z + 1)^2 + 9 = 0$$

precisely when $(z + 1)^2 = -9$; in other words, precisely when $z + 1 = \pm 3i$, so that $z = -1 \pm 3i$.

PROBLEMS FOR CHAPTER 9

1. Find each of the following:

a) $(1 - 2i)^2$

b) $(1 + 2i)(3 + 4i)$

c) $\frac{1}{3 + 2i}$

d) $\frac{(1 - 2i)^2}{1 - i}$

e) $\frac{1}{1 + i} + \frac{1}{1 - 2i}$

f) $i^3 + i^4 + i^5 + i^6$

g) $\frac{1}{(4 + 2i)(2 - 3i)}$

h) $\frac{3 + 4i}{1 + 2i}$

2. Find the real and imaginary parts of each of the complex numbers in Question 1.

3. Solve each of the following equations:

a) $(2 + i)z + i = 3$

b) $\frac{z - 1}{z - i} = \frac{2}{3}$

c) $\frac{z - 3i}{z + 4} = \frac{1}{5}$

d) $\frac{1}{z + i} = \frac{3}{2 - z}$

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