ELEMENTARY MATHEMATICS

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Chapter 14

INTRODUCTION TO INTEGRATION

14.1. Antiderivatives

In this chapter, we discuss the inverse process of differentiation. In other words, given a function $f(x)$, we wish to find a function $F(x)$ such that $F'(x) = f(x)$. Any such function $F(x)$ is called an antiderivative, or indefinite integral, of the function $f(x)$, and we write

$$F(x) = \int f(x)\,\mathrm{d}x.$$

A first observation is that the antiderivative, if it exists, is not unique. Suppose that the function $F(x)$ is an antiderivative of the function $f(x)$, so that $F'(x) = f(x)$. Let $G(x) = F(x) + C$, where C is any fixed real number. Then it is easy to see that $G'(x) = F'(x) = f(x)$, so that $G(x)$ is also an antiderivative of $f(x)$. A second observation, somewhat less obvious, is that for any given function $f(x)$, any two distinct antiderivatives of $f(x)$ must differ only by a constant. In other words, if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x)$ is a constant. In this chapter, we shall denote any such constant by C, with or without subscripts.

An immediate consequence of this second observation is the following simple result related to the derivatives of constants in Section 11.1.

ANTIDERIVATIVES OF ZERO. *We have*

$$\int 0\,\mathrm{d}x = C.$$

In other words, the antiderivatives of the zero function are precisely all the constant functions.

Indeed, many antiderivatives can be obtained simply by referring to various rules concerning derivatives. We list here a number of such results. The first of these is related to the constant multiple rule for differentiation in Section 11.2.

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† This chapter was written at Macquarie University in 1999.

CONSTANT MULTIPLE RULE. *Suppose that a function $f(x)$ has antiderivatives. Then for any fixed real number c, we have*

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x.$$

ANTIDERIVATIVES OF POWERS.

(a) Suppose that n is a fixed real number such that $n \neq -1$. Then

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C.$$

(b) We have

$$\int x^{-1}\,\mathrm{d}x = \log|x| + C.$$

PROOF. Part (a) is a consequence of the rule concerning derivatives of powers in Section 11.1. If $x > 0$, then part (b) is a consequence of the rule concerning the derivative of the logarithmic function in Section 12.3. If $x < 0$, we can write $|x| = u$, where $u = -x$. It then follows from the Chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log|x|) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}u}(\log u) = -\frac{1}{u} = \frac{1}{x} \tag{1}$$

again. ♣

Corresponding to the sum rule for differentiation in Section 11.2, we have the following.

SUM RULE. *Suppose that functions $f(x)$ and $g(x)$ have antiderivatives. Then*

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

We next consider trigonometric functions.

ANTIDERIVATIVES OF TRIGONOMETRIC FUNCTIONS.

(a) We have

$$\int \cos x\,\mathrm{d}x = \sin x + C \qquad \textit{and} \qquad \int \sin x\,\mathrm{d}x = -\cos x + C.$$

(b) We have

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C \qquad \textit{and} \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C.$$

(c) We have

$$\int \tan x \sec x\,\mathrm{d}x = \sec x + C \qquad \textit{and} \qquad \int \cot x \csc x\,\mathrm{d}x = -\csc x + C.$$

(d) We have

$$\int \sec x\,\mathrm{d}x = \log|\tan x + \sec x| + C \qquad \textit{and} \qquad \int \csc x\,\mathrm{d}x = -\log|\cot x + \csc x| + C.$$

PROOF. Parts (a)–(c) follow immediately from the rules concerning derivatives of the trigonometric functions in Section 11.3. Part (d) follows from Example 12.3.12 and Example 12.3.13 if we note (1). ♣

Corresponding to the rule concerning the derivative of the exponential function in Section 12.3, we have the following.

ANTIDERIVATIVES OF THE EXPONENTIAL FUNCTION. *We have*

$$\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C.$$

EXAMPLE 14.1.1. Using the sum rule, the constant multiple rule and the rule concerning antiderivatives of powers, we have

$$\int (x^2 + 3x + 1) \,\mathrm{d}x = \int x^2 \,\mathrm{d}x + 3\int x \,\mathrm{d}x + \int x^0 \,\mathrm{d}x = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C.$$

EXAMPLE 14.1.2. Using the sum rule and the rules concerning antiderivatives of powers and of trigonometric functions, we have

$$\int (x^3 + \sin x) \,\mathrm{d}x = \int x^3 \,\mathrm{d}x + \int \sin x \,\mathrm{d}x = \frac{1}{4}x^4 - \cos x + C.$$

EXAMPLE 14.1.3. We have

$$\int (\sin x + \sec x) \,\mathrm{d}x = \int \sin x \,\mathrm{d}x + \int \sec x \,\mathrm{d}x = -\cos x + \log|\tan x + \sec x| + C.$$

EXAMPLE 14.1.4. We have

$$\int (\mathrm{e}^x + 3\cos x) \,\mathrm{d}x = \int \mathrm{e}^x \,\mathrm{d}x + 3\int \cos x \,\mathrm{d}x = \mathrm{e}^x + 3\sin x + C.$$

EXAMPLE 14.1.5. To find

$$\int \frac{1 - \sin x}{1 + \sin x} \,\mathrm{d}x,$$

note first of all that

$$\begin{aligned} \frac{1 - \sin x}{1 + \sin x} &= \frac{(1 - \sin x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{1 - 2\sin x + \sin^2 x}{1 - \sin^2 x} = \frac{1 - 2\sin x + \sin^2 x}{\cos^2 x} \\ &= \sec^2 x - 2\tan x \sec x + \tan^2 x = 2\sec^2 x - 2\tan x \sec x - 1. \end{aligned}$$

It follows that

$$\int \frac{1 - \sin x}{1 + \sin x} \,\mathrm{d}x = 2\int \sec^2 x \,\mathrm{d}x - 2\int \tan x \sec x \,\mathrm{d}x - \int \mathrm{d}x = 2\tan x - 2\sec x - x + C.$$

14.2. Integration by Substitution

We now discuss how we can use the chain rule in differentiation to help solve problems in integration. This technique is usually called integration by substitution. As we shall not prove any result here, our discussion will be only heuristic.

We emphasize that the technique does not always work. First of all, we have little or no knowledge of the antiderivatives of many functions. Secondly, there is no simple routine that we can describe to help us find a suitable substitution even in the cases where the technique works. On the other hand, when the technique does work, there may well be more than one suitable substitution!

REMARK. It is imperative that one does not give up when one's effort does not seem to yield results. We learn far more from indefinite integrals that we cannot find than from those than we can.

INTEGRATION BY SUBSTITUTION – VERSION 1. *If we make a substitution $x = g(u)$, then $\mathrm{d}x = g'(u)\,\mathrm{d}u$, and*

$$\int f(x)\,\mathrm{d}x = \int f(g(u))g'(u)\,\mathrm{d}u.$$

Example 14.2.1. Consider the indefinite integral

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

If we make a substitution $x = \sin u$, then $\mathrm{d}x = \cos u\,\mathrm{d}u$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \sin^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cos v$, then $\mathrm{d}x = -\sin v\,\mathrm{d}v$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{\sin v}{\sqrt{1-\cos^2 v}}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cos^{-1} x + C.$$

See Section 12.4 concerning derivatives of inverse trigonometric functions.

Example 14.2.2. Consider the indefinite integral

$$\int \frac{1}{1+x^2}\,\mathrm{d}x.$$

If we make a substitution $x = \tan u$, then $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \tan^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cot v$, then $\mathrm{d}x = -\csc^2 v\,\mathrm{d}v$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\int \frac{\csc^2 v}{1+\cot^2 v}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cot^{-1} x + C.$$

Example 14.2.3. Consider the indefinite integral

$$\int x\sqrt{x+1}\,\mathrm{d}x.$$

If we make a substitution $x = u^2 - 1$, then $\mathrm{d}x = 2u\,\mathrm{d}u$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int 2(u^2-1)u^2\,\mathrm{d}u = 2\int u^4\,\mathrm{d}u - 2\int u^2\,\mathrm{d}u \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

On the other hand, if we make a substitution $x = v - 1$, then $\mathrm{d}x = \mathrm{d}v$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (v-1)v^{1/2}\,\mathrm{d}v = \int v^{3/2}\,\mathrm{d}v - \int v^{1/2}\,\mathrm{d}v \\ &= \frac{2}{5}v^{5/2} - \frac{2}{3}v^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

We can confirm that the indefinite integral is correct by checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C\right) = x\sqrt{x+1}.$$

INTEGRATION BY SUBSTITUTION – VERSION 2. *Suppose that a function $f(x)$ can be written in the form $f(x)=g(h(x))h'(x)$. If we make a substitution $u=h(x)$, then $\mathrm{d}u=h'(x)\,\mathrm{d}x$, and*

$$\int f(x)\,\mathrm{d}x=\int g(h(x))h'(x)\,\mathrm{d}x=\int g(u)\,\mathrm{d}u.$$

REMARK. Note that in Version 1, the variable x is initially written as a function of the new variable u, whereas in Version 2, the new variable u is written as a function of x. The difference, however, is minimal, as the substitution $x=g(u)$ in Version 1 has to be invertible to enable us to return from the new variable u to the original variable x at the end of the process.

EXAMPLE 14.2.4. Consider the indefinite integral

$$\int x(x^2+3)^4\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^2+3 is equal to $2x$, so it is convenient to make the substitution $u=x^2+3$. Then $\mathrm{d}u=2x\,\mathrm{d}x$, and

$$\int x(x^2+3)^4\,\mathrm{d}x=\frac{1}{2}\int 2x(x^2+3)^4\,\mathrm{d}x=\frac{1}{2}\int u^4\,\mathrm{d}u=\frac{1}{10}u^5+C=\frac{1}{10}(x^2+3)^5+C.$$

EXAMPLE 14.2.5. Consider the indefinite integral

$$\int \frac{1}{x\log x}\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\log x$ is equal to $1/x$, so it is convenient to make the substitution $u=\log x$. Then $\mathrm{d}u=(1/x)\,\mathrm{d}x$, and

$$\int \frac{1}{x\log x}\,\mathrm{d}x=\int\frac{1}{u}\,\mathrm{d}u=\log|u|+C=\log|\log x|+C.$$

EXAMPLE 14.2.6. Consider the indefinite integral

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^3 is equal to $3x^2$, so it is convenient to make the substitution $u=x^3$. Then $\mathrm{d}u=3x^2\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int \mathrm{e}^u\,\mathrm{d}u=\frac{1}{3}\mathrm{e}^u+C=\frac{1}{3}\mathrm{e}^{x^3}+C.$$

A somewhat more complicated alternative is to note that the derivative of the function e^{x^3} is equal to $3x^2\mathrm{e}^{x^3}$, so it is convenient to make the substitution $v=\mathrm{e}^{x^3}$. Then $\mathrm{d}v=3x^2\mathrm{e}^{x^3}\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int \mathrm{d}v=\frac{1}{3}v+C=\frac{1}{3}\mathrm{e}^{x^3}+C.$$

EXAMPLE 14.2.7. Consider the indefinite integral

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\tan x$ is equal to $\sec^2 x$, so it is convenient to make the substitution $u=\tan x$. Then $\mathrm{d}u=\sec^2 x\,\mathrm{d}x$, and

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x=\int u^3\,\mathrm{d}u=\frac{1}{4}u^4+C=\frac{1}{4}\tan^4 x+C.$$

Occasionally, the possibility of substitution may not be immediately obvious, and a certain amount of trial and error does occur. The fact that one substitution does not appear to work does not mean that the method fails. It may very well be the case that we have used a bad substitution. Or perhaps we may slightly modify the problem first. We illustrate this point by looking at two more examples.

Example 14.2.8. Consider the indefinite integral

$$\int \tan x \, \mathrm{d}x.$$

Here it does not appear that any substitution will work. However, if we write

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x,$$

then we observe that the derivative of the function $\cos x$ is equal to $-\sin x$, so it is convenient to make the substitution $u = \cos x$. Then $\mathrm{d}u = -\sin x \, \mathrm{d}x$, and

$$\int \tan x \, \mathrm{d}x = -\int \frac{-\sin x}{\cos x} \, \mathrm{d}x = -\int \frac{1}{u} \, \mathrm{d}u = -\log |u| + C = -\log |\cos x| + C.$$

Example 14.2.9. The indefinite integral

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \, \mathrm{d}x$$

is rather daunting at first sight, but we have enough technique to study it. Note first of all that

$$\begin{aligned} 9 + 6x + 2x^2 + x^3 &= 9 + 2x + 2x^2 + 4x + x^3 = 9 + 2x + 2x^2 + x(4 + x^2) \\ &= 1 + 2x + 8 + 2x^2 + x(4 + x^2) = 1 + 2x + 2(4 + x^2) + x(4 + x^2). \end{aligned}$$

It follows that

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \, \mathrm{d}x = \int \frac{1}{4 + x^2} \, \mathrm{d}x + \int \frac{2x}{4 + x^2} \, \mathrm{d}x + \int (2 + x) \, \mathrm{d}x. \tag{2}$$

To study the first integral on the right hand side of (2), we can make a substitution $x = 2\tan u$. Then $\mathrm{d}x = 2\sec^2 u \, \mathrm{d}u$, and

$$\int \frac{1}{4 + x^2} \, \mathrm{d}x = \int \frac{2\sec^2 u}{4 + 4\tan^2 u} \, \mathrm{d}u = \frac{1}{2} \int \mathrm{d}u = \frac{1}{2} u + C_1 = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C_1. \tag{3}$$

To study the second integral on the right hand side of (2), we note that the derivative of the function $4 + x^2$ is equal to $2x$. If we make a substitution $v = 4 + x^2$, then $\mathrm{d}v = 2x \, \mathrm{d}x$, and

$$\int \frac{2x}{4 + x^2} \, \mathrm{d}x = \int \frac{1}{v} \, \mathrm{d}v = \log |v| + C_2 = \log(4 + x^2) + C_2. \tag{4}$$

The third integral on the right hand side of (2) is easy to evaluate. We have

$$\int (2 + x) \, \mathrm{d}x = 2x + \frac{1}{2} x^2 + C_3. \tag{5}$$

Substituting (3)–(5) into (2) and writing $C = C_1 + C_2 + C_3$, we obtain

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \, \mathrm{d}x = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \log(4 + x^2) + 2x + \frac{1}{2} x^2 + C.$$

It may be worth checking that

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \log(4 + x^2) + 2x + \frac{1}{2} x^2 + C \right) = \frac{9 + 6x + 2x^2 + x^3}{4 + x^2}.$$

14.3. Definite Integrals

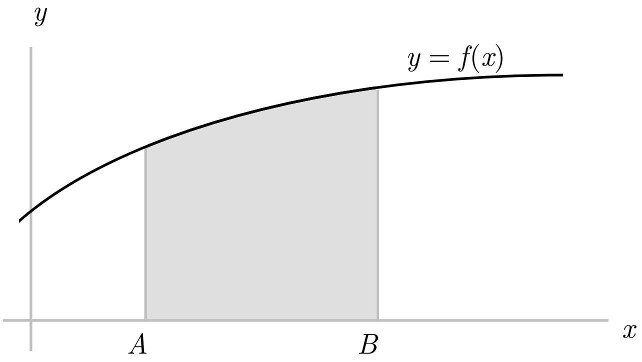
The formal definition of a definite integral is rather complicated, and we do not propose to discuss it here. Instead, we shall only give some geometric motivation, and then relate the definite integral to indefinite integrals we have discussed earlier.

Suppose that $f(x)$ is a real valued function, defined on an interval $[A,B]=\{x\in\mathbb{R}:A\le x\le B\}$. We shall suppose also that $f(x)$ has an antiderivative $F(x)$ for every $x\in[A,B]$.

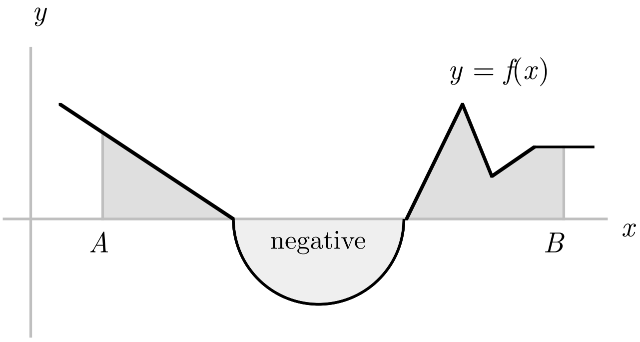
Consider first of all the special case that $f(x)\ge 0$ for every $x\in[A,B]$. By the definite integral

$$\int_A^B f(x)\,\mathrm{d}x,$$

we mean the area below the curve $y=f(x)$ and above the horizontal axis $y=0$, bounded between the vertical lines $x=A$ and $x=B$, as shown in the picture below.



In general, we take the area between the curve $y=f(x)$ and the horizontal axis $y=0$, bounded between the vertical lines $x=A$ and $x=B$, with the convention that the area below the horizontal axis $y=0$ is taken to be negative, as shown in the picture below.



We now need a way of calculating this area. In some very special cases, this is very simple.

Example 14.3.1. If we examine the graph of the trigonometric functions in Chapter 3, then it is easy to see that

$$\int_0^{2\pi}\sin x\,\mathrm{d}x=0\qquad\text{and}\qquad\int_0^{\pi}\cos x\,\mathrm{d}x=0.$$

In each case, it is easy to see that the area in question above the horizontal axis $y=0$ is equal to the area in question below this axis.

Example 14.3.2. It is easy to see that the area between the line $y = x$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = 0$ and $x = 1$, is the area of a triangle with base 1 and height 1. Hence

$$\int_0^1 x \,\mathrm{d}x = \frac{1}{2}.$$

In many instances, we do not have such geometric information to help us calculate the area in question. Instead, we can use the indefinite integral.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. *Suppose that a function $F(x)$ satisfies $F'(x) = f(x)$ for every $x \in [A, B]$. Then*

$$\int_A^B f(x) \,\mathrm{d}x = \Big[F(x) \Big]_A^B = F(B) - F(A).$$

Remark. A simple consequence of the above is that the constant multiple rule and sum rule for indefinite integrals extend to definite integrals. For any fixed real number c, we have

$$\int_A^B cf(x) \,\mathrm{d}x = c \int_A^B f(x) \,\mathrm{d}x.$$

We also have

$$\int_A^B (f(x) + g(x)) \,\mathrm{d}x = \int_A^B f(x) \,\mathrm{d}x + \int_A^B g(x) \,\mathrm{d}x.$$

A further consequence of the Fundamental theorem of integral calculus is a rule concerning splitting up an interval $[A, B]$ into two. Suppose that $A < A^* < B$. Then

$$\int_A^B f(x) \,\mathrm{d}x = \int_A^{A^*} f(x) \,\mathrm{d}x + \int_{A^*}^B f(x) \,\mathrm{d}x.$$

Example 14.3.3. Returning to Example 14.3.1, we have

$$\int_0^{2\pi} \sin x \,\mathrm{d}x = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0 = 0$$

and

$$\int_0^{\pi} \cos x \,\mathrm{d}x = \Big[\sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0.$$

Example 14.3.4. Returning to Example 14.3.2, we have

$$\int_0^1 x \,\mathrm{d}x = \left[\frac{1}{2} x \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 14.3.5. We have

$$\int_0^{\pi} \sin x \,\mathrm{d}x = \Big[-\cos x \Big]_0^{\pi} = -\cos \pi + \cos 0 = 2.$$

Example 14.3.6. We have

$$\int_1^2 \frac{1}{x} \,\mathrm{d}x = \Big[\log |x| \Big]_1^2 = \log 2 - \log 1 = \log 2.$$

EXAMPLE 14.3.7. We have

$$\int_0^1 \mathrm{e}^x \,\mathrm{d}x = \Big[\mathrm{e}^x\Big]_0^1 = \mathrm{e}^1 - \mathrm{e}^0 = \mathrm{e} - 1.$$

EXAMPLE 14.3.8. We have

$$\int_0^{\pi/4} \sec^2 x \,\mathrm{d}x = \Big[\tan x\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1.$$

EXAMPLE 14.3.9. We have

$$\int_{-1}^1 (x^3 + x^2)\,\mathrm{d}x = \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^1 = \left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right) = \frac{2}{3}.$$

EXAMPLE 14.3.10. Recall Example 14.2.1. Since

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C, \tag{6}$$

we have

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \Big[\sin^{-1} x\Big]_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$$

To obtain (6), recall that we can use the substitution $x = \sin u$ to show that

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \mathrm{d}u = u + C,$$

followed by an inverse substitution $u = \sin^{-1} x$. Here, we need to make the extra step of substituting the values $x = 0$ and $x = 1/2$ to the indefinite integral $\sin^{-1} x$. Observe, however, that with the substitution $x = \sin u$, the variable x increases from 0 to $1/2$ as the variable u increases from 0 to $\pi/6$. But then

$$\int_0^{\pi/6} \mathrm{d}u = \Big[u\Big]_0^{\pi/6} = \frac{\pi}{6} = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x,$$

so it appears that we do not need the inverse substitution $u = \sin^{-1} x$. Perhaps we can directly substitute $u = 0$ and $u = \pi/6$ to the indefinite integral u.

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 1. *Suppose that a substitution $x = g(u)$ satisfies the following conditions:*
(a) There exist $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) = A$ and $g(\beta) = B$.
(b) The derivative $g'(u) > 0$ for every u satisfying $\alpha < u < \beta$.
Then $\mathrm{d}x = g'(u)\,\mathrm{d}u$, and

$$\int_A^B f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(u))g'(u)\,\mathrm{d}u.$$

REMARK. If condition (b) above is replaced by the condition that the derivative $g'(u) < 0$ for every u satisfying $\beta < u < \alpha$, then the same conclusion holds if we adopt the convention that

$$\int_\alpha^\beta f(g(u))g'(u)\,\mathrm{d}u = -\int_\beta^\alpha f(g(u))g'(u)\,\mathrm{d}u.$$

Example 14.3.11. To calculate the definite integral

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x,$$

we can use the substitution $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Note that $\tan 0 = 0$ and $\tan(\pi/4) = 1$, and that $\sec^2 u > 0$ whenever $0 < u < \pi/4$. It follows that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \int_0^{\pi/4} \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int_0^{\pi/4} \mathrm{d}u = \Big[u\Big]_0^{\pi/4} = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

We can compare this to first observing Example 14.2.2, so that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Example 14.3.12. To calculate the definite integral

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x,$$

we can use the substitution $x = g(u) = u^2 - 1$, so that $\mathrm{d}x = 2u\,\mathrm{d}u$. Note that $g(1) = 0$ and $g(2) = 3$, and that $g'(u) = 2u > 0$ whenever $1 < u < 2$. It follows that

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x = \int_1^2 2(u^2-1)u^2\,\mathrm{d}u = \left[\frac{2}{5}u^5 - \frac{2}{3}u^3\right]_1^2 = \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{62}{5} - \frac{14}{3} = \frac{116}{15}.$$

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 2. *Suppose that a substitution* $u = h(x)$ *satisfies the following conditions:*

(a) There exists a function $g(u)$ *such that* $f(x) = g(h(x))h'(x)$ *for every* $x \in [A, B]$.

(b) The derivative $h'(x) > 0$ *for every* x *satisfying* $A < x < B$.

Then $\mathrm{d}u = h'(x)\,\mathrm{d}x$, *and*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^B g(h(x))h'(x)\,\mathrm{d}x = \int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u.$$

Remark. If condition (b) above is replaced by the condition that the derivative $h'(x) < 0$ for every x satisfying $A < x < B$, then the same conclusion holds if we adopt the convention that

$$\int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u = -\int_{h(B)}^{h(A)} g(u)\,\mathrm{d}u.$$

Example 14.3.13. To calculate the definite integral

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = x^2 + 3$, so that $\mathrm{d}u = 2x\,\mathrm{d}x$. Note that $h(0) = 3$ and $h(1) = 4$, and that $h'(x) = 2x > 0$ whenever $0 < x < 1$. It follows that

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int_3^4 u^4\,\mathrm{d}u = \frac{1}{2}\left[\frac{u^5}{5}\right]_3^4 = \frac{1}{2}\left(\frac{1024}{5} - \frac{243}{5}\right) = \frac{781}{10}.$$

Example 14.3.14. To calculate the definite integral

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \log x$, so that $\mathrm{d}u = h'(x)\,\mathrm{d}x$, where $h'(x) = 1/x > 0$ whenever $2 < x < 4$. Note also that $h(2) = \log 2$ and $h(4) = \log 4$. It follows that

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x = \int_{\log 2}^{\log 4} \frac{1}{u}\,\mathrm{d}u = \Big[\log|u|\Big]_{\log 2}^{\log 4} = \log\log 4 - \log\log 2 = \log\left(\frac{\log 4}{\log 2}\right) = \log 2.$$

Example 14.3.15. To calculate the definite integral

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$. Now $h(0) = 0$ and $h(\pi) = 0$, so something is funny here! The problem is that

$$h'(x) = \cos x \begin{cases} > 0 & \left(0 < x < \frac{\pi}{2}\right), \\ < 0 & \left(\frac{\pi}{2} < x < \pi\right). \end{cases}$$

It follows that we must first write

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x + \int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x \tag{7}$$

before we can make any substitution. Consider now the first integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(0) = 0$ and $h(\pi/2) = 1$, and that $h'(x) > 0$ whenever $0 < x < \pi/2$. Hence

$$\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x = \int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}.$$

Consider next the second integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(\pi/2) = 1$ and $h(\pi) = 0$, and that $h'(x) < 0$ whenever $\pi/2 < x < \pi$. Hence

$$\int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_1^0 u^2\,\mathrm{d}u = -\int_0^1 u^2\,\mathrm{d}u = -\frac{1}{3}.$$

Combining the two parts, we conclude that

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = 0.$$

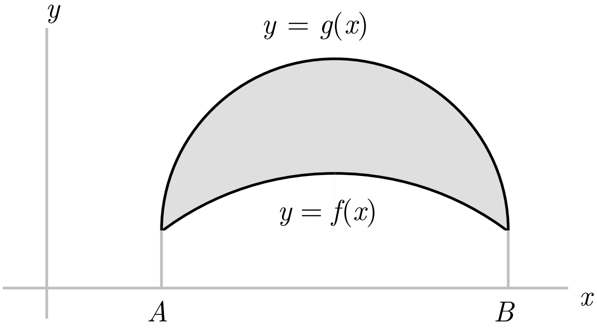
Alternatively, we can make the substitution $x = g(v) = \pi - v$ to the second integral on the right hand side of (7). Then $g(\pi/2) = \pi/2$ and $g(0) = \pi$, and $g'(v) = -1 < 0$ for every v satisfying $0 < v < \pi/2$. It follows that

$$\int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_{\pi/2}^0 \sin^2 v\cos v\,\mathrm{d}v = -\int_0^{\pi/2} \sin^2 v\cos v\,\mathrm{d}v.$$

This, combined with (7), gives the same conclusion.

14.4. Areas

We conclude this chapter by describing how we may use definite integrals to evaluate areas. Suppose that the boundary of a region on the xy-plane can be described by a top edge $y = g(x)$ and a bottom edge $y = f(x)$ bounded between two vertical lines $x = A$ and $x = B$, as shown in the picture below.



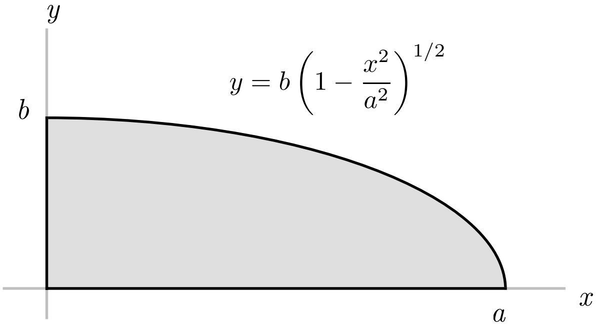
Then the area of the region is given by the definite integral

$$\int_A^B (g(x) - f(x))\,\mathrm{d}x.$$

Example 14.4.1. We wish to show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b \in \mathbb{R}$ are positive, is equal to πab. To do this, we may consider the quarter of the ellipse in the first quadrant, as shown in the picture below.



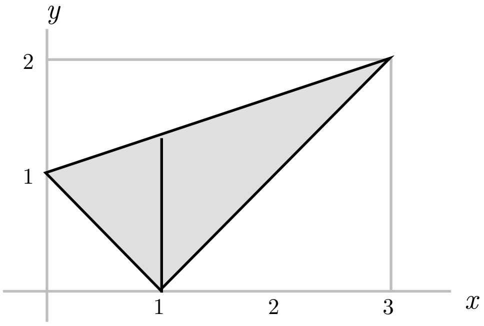
It follows that the shaded region has area

$$\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2}\,\mathrm{d}x.$$

We can use the substitution $x = g(u) = a\sin u$. Then $g(0) = 0$ and $g(\pi/2) = a$. Furthermore, we have $\mathrm{d}x = g'(u)\,\mathrm{d}u$, where $g'(u) = a\cos u > 0$ whenever $0 < u < \pi/2$. It follows that

$$\begin{aligned}\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2}\,\mathrm{d}x &= \int_0^{\pi/2} ab(1 - \sin^2 u)^{1/2}\cos u\,\mathrm{d}u = ab\int_0^{\pi/2}\cos^2 u\,\mathrm{d}u \\ &= ab\int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)\mathrm{d}u = ab\left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi ab}{4}.\end{aligned}$$

Example 14.4.2. We wish to evaluate the area of the triangle with vertices $(0,1)$, $(1,0)$ and $(3,2)$. To do this, we split the triangle into two regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x=0$ and $x=1$, and the top edge and the bottom edge are given respectively by

$$y=\frac{1}{3}x+1 \qquad \text{and} \qquad y=1-x.$$

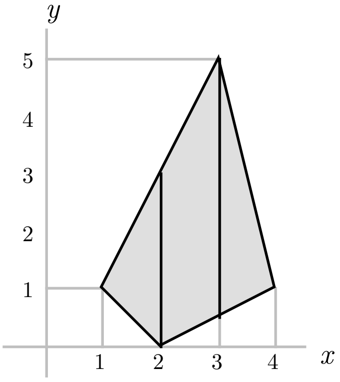
The triangle on the right is bounded between the vertical lines $x=1$ and $x=3$, and the top edge and the bottom edge are given respectively by

$$y=\frac{1}{3}x+1 \qquad \text{and} \qquad y=x-1.$$

It follows that the area of the original triangle is given by

$$\begin{aligned}&\int_0^1 \left(\left(\frac{1}{3}x+1\right)-(1-x)\right)\,\mathrm{d}x+\int_1^3 \left(\left(\frac{1}{3}x+1\right)-(x-1)\right)\,\mathrm{d}x\\ &\quad=\int_0^1 \frac{4}{3}x\,\mathrm{d}x+\int_1^3 \left(2-\frac{2}{3}x\right)\,\mathrm{d}x=\left[\frac{2}{3}x^2\right]_0^1+\left[2x-\frac{1}{3}x^2\right]_1^3=2.\end{aligned}$$

Example 14.4.3. We wish to evaluate the area of the quadrilateral with vertices $(1,1)$, $(2,0)$, $(4,1)$ and $(3,5)$. To do this, we split the quadrilateral into three regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x=1$ and $x=2$, and the top edge and the bottom edge are given respectively by

$$y=2x-1 \qquad \text{and} \qquad y=2-x.$$

The quadrilateral in the middle is bounded between the vertical lines $x = 2$ and $x = 3$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

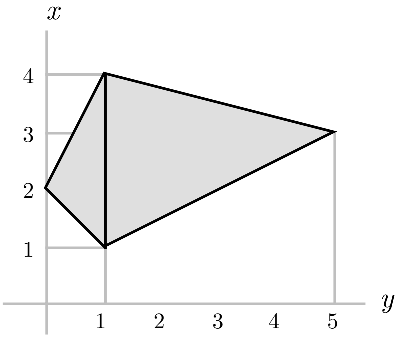
The triangle on the right is bounded between the vertical lines $x = 3$ and $x = 4$, and the top edge and the bottom edge are given respectively by

$$y = 17 - 4x \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_1^2 ((2x-1)-(2-x))\,\mathrm{d}x + \int_2^3 \left((2x-1) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x + \int_3^4 \left((17-4x) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x \\
&\quad = \int_1^2 (3x-3)\,\mathrm{d}x + \int_2^3 \frac{3}{2}x\,\mathrm{d}x + \int_3^4 \left(18 - \frac{9}{2}x\right)\mathrm{d}x \\
&\quad = \left[\frac{3}{2}x^2 - 3x\right]_1^2 + \left[\frac{3}{4}x^2\right]_2^3 + \left[18x - \frac{9}{4}x^2\right]_3^4 = \frac{15}{2}.
\end{aligned}$$

Alternatively, we can transpose the picture above and split the quadrilateral into two regions as shown in the picture below:



Note that the roles of x and y are now interchanged. The triangle on the left is bounded between the vertical lines $y = 0$ and $y = 1$, and the top edge and the bottom edge are given respectively by

$$x = 2y + 2 \qquad \text{and} \qquad x = 2 - y.$$

The triangle on the right is bounded between the vertical lines $y = 1$ and $y = 5$, and the top edge and the bottom edge are given respectively by

$$x = \frac{17}{4} - \frac{1}{4}y \qquad \text{and} \qquad x = \frac{1}{2}y + \frac{1}{2}.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_0^1 ((2y+2)-(2-y))\,\mathrm{d}y + \int_1^5 \left(\left(\frac{17}{4} - \frac{1}{4}y\right) - \left(\frac{1}{2}y + \frac{1}{2}\right)\right)\mathrm{d}y \\
&\quad = \int_0^1 3y\,\mathrm{d}y + \int_1^5 \left(\frac{15}{4} - \frac{3}{4}y\right)\mathrm{d}y = \left[\frac{3}{2}y^2\right]_0^1 + \left[\frac{15}{4}y - \frac{3}{8}y^2\right]_1^5 = \frac{15}{2}
\end{aligned}$$

as before.

PROBLEMS FOR CHAPTER 14

1. Find each of the following indefinite integrals:

a) $\displaystyle\int \sqrt{3}\,\mathrm{d}x$ b) $\displaystyle\int (5x+3)\,\mathrm{d}x$ c) $\displaystyle\int (2x^2-3x+1)\,\mathrm{d}x$

d) $\displaystyle\int x^3\,\mathrm{d}x$ e) $\displaystyle\int (x-2)(x+3)\,\mathrm{d}x$ f) $\displaystyle\int (1-2\cos x)\,\mathrm{d}x$

g) $\displaystyle\int (5\cos x+4x)\,\mathrm{d}x$ h) $\displaystyle\int 8\mathrm{e}^x\,\mathrm{d}x$ i) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x$

2. Evaluate each of the following indefinite integrals using the given substitution:

a) $\displaystyle\int x(x^2-1)^{99}\,\mathrm{d}x$ (use the substitution $u=x^2-1$)

b) $\displaystyle\int \frac{x^2}{\sqrt{2+x^3}}\,\mathrm{d}x$ (use the substitution $u=x^3+2$)

c) $\displaystyle\int \sin 4x\,\mathrm{d}x$ (use the substitution $u=4x$)

d) $\displaystyle\int \frac{\mathrm{d}x}{(2x+1)^2}$ (use the substitution $u=2x+1$)

e) $\displaystyle\int \frac{x+3}{(x^2+6x)^2}\,\mathrm{d}x$ (use the substitution $u=x^2+6x$)

f) $\displaystyle\int \sec ax\,\tan ax\,\mathrm{d}x$ (use the substitution $u=ax$)

3. Evaluate each of the following indefinite integrals:

a) $\displaystyle\int \cos 2x\,\mathrm{d}x$ b) $\displaystyle\int \sqrt{x-1}\,\mathrm{d}x$ c) $\displaystyle\int x^2\cos(1-x^3)\,\mathrm{d}x$

d) $\displaystyle\int x\sin(x^2)\,\mathrm{d}x$ e) $\displaystyle\int \frac{1}{(1-3x)^4}\,\mathrm{d}x$ f) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$

g) $\displaystyle\int \sec^2(3x)\,\mathrm{d}x$ h) $\displaystyle\int \sin^3 x\cos x\,\mathrm{d}x$ i) $\displaystyle\int x(x^2+16)^2\,\mathrm{d}x$

j) $\displaystyle\int x^2\sqrt{x^3+8}\,\mathrm{d}x$ k) $\displaystyle\int \frac{1}{\sqrt{2x+5}}\,\mathrm{d}x$ l) $\displaystyle\int \left(x-\frac{1}{x}\right)\mathrm{d}x$

m) $\displaystyle\int \frac{2x+1}{x^2+x+3}\,\mathrm{d}x$ n) $\displaystyle\int \frac{1}{x^2-4x+4}\,\mathrm{d}x$ o) $\displaystyle\int \frac{\log x}{x}\,\mathrm{d}x$

p) $\displaystyle\int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x$ q) $\displaystyle\int x\mathrm{e}^{x^2}\,\mathrm{d}x$ r) $\displaystyle\int \mathrm{e}^{2x-1}\,\mathrm{d}x$

s) $\displaystyle\int \sec(4x)\tan(4x)\,\mathrm{d}x$ t) $\displaystyle\int x^3\cos(5x^4)\,\mathrm{d}x$ u) $\displaystyle\int \sec^2(2x+1)\,\mathrm{d}x$

v) $\displaystyle\int \mathrm{e}^x\cos(\mathrm{e}^x)\,\mathrm{d}x$ w) $\displaystyle\int \frac{(\log x)^2}{x}\,\mathrm{d}x$ x) $\displaystyle\int \tan x\sec^3 x\,\mathrm{d}x$

4. Evaluate each of the following definite integrals:

a) $\displaystyle\int_1^2 2x\,\mathrm{d}x$ b) $\displaystyle\int_1^3 \frac{1}{x}\,\mathrm{d}x$ c) $\displaystyle\int_0^2 \mathrm{e}^{-x}\,\mathrm{d}x$

d) $\displaystyle\int_2^3 (3x+1)\,\mathrm{d}x$ e) $\displaystyle\int_0^{\pi} \sin x\,\mathrm{d}x$ f) $\displaystyle\int_3^6 (x-3)^2\,\mathrm{d}x$

g) $\displaystyle\int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x$ h) $\displaystyle\int_0^1 x\mathrm{e}^{x^2}\,\mathrm{d}x$ i) $\displaystyle\int_0^a (x^2+a^2)\,\mathrm{d}x$

j) $\displaystyle\int_0^1 (1+x+3x^2)\,\mathrm{d}x$ k) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$ l) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,\mathrm{d}x$

5. a) Draw the graphs of the line $y = x$ and the parabola $y = x^2$.
   b) Find the two points of intersection of the two curves.
   c) Use definite integrals to find the area bounded between the two curves.

6. Use definite integrals to find the area between the curves $y = \mathrm{e}^x$ and $y = \mathrm{e}^{2x}$, bounded between the lines $x = 0$ and $x = 1$.

7. Find the area of the triangle with vertices $(0, 0)$, $(4, 3)$ and $(1, 5)$.

8. Find the area of the quadrilateral with vertices $(1, 1)$, $(5, 2)$, $(2, 3)$ and $(4, 3)$.

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