INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 7

LEBESGUE INTEGRALS ON UNBOUNDED INTERVALS

7.1. Some Limiting Cases

We begin by considering the following result which extends Lebesgue integrals on finite intervals to infinite intervals.

THEOREM 7A. *Suppose that $I=[A,\infty)$, where $A\in\mathbb{R}$. Suppose further that the function $f:I\to\mathbb{R}$ satisfies the following conditions:*
(a) $f\in\mathcal{L}([A,B])$ for every real number $B\ge A$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)|\,\mathrm{d}x\le M$ for every real number $B\ge A$.
Then $f\in\mathcal{L}(I)$, the limit

$$\lim_{B\to\infty}\int_A^B f(x)\,\mathrm{d}x$$

exists, and

$$\int_A^\infty f(x)\,\mathrm{d}x=\lim_{B\to\infty}\int_A^B f(x)\,\mathrm{d}x. \tag{1}$$

Proof. Let $B_n\in\mathbb{R}$ be an increasing sequence satisfying $B_n\ge A$ for every $n\in\mathbb{N}$ and $B_n\to\infty$ as $n\to\infty$. For every $n\in\mathbb{N}$, define $f_n:I\to\mathbb{R}$ by writing

$$f_n(x)=\begin{cases} f(x) & \text{if } x\in[A,B_n],\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $f_n \in \mathcal{L}(I)$, in view of Theorem 4J. Furthermore, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in I$, and so $|f_n(x)| \to |f(x)|$ as $n \to \infty$ for every $x \in I$. It is not difficult to see that the sequence $|f_n|$ is increasing on I, so that

$$\int_I |f_n(x)| \, \mathrm{d}x$$

is an increasing sequence, bounded above by M in view of (b), and so converges as $n \to \infty$. It follows from the Monotone convergence theorem (Theorem 5C) that $|f| \in \mathcal{L}(I)$. Note also that $|f_n(x)| \le |f(x)|$ for every $x \in I$. It follows from the Dominated convergence theorem (Theorem 6A) that $f \in \mathcal{L}(I)$, and that

$$\int_I f(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_A^{B_n} f(x) \, \mathrm{d}x.$$

Note that this holds for every increasing sequence $B_n \to \infty$ as $n \to \infty$, and so the equality (1) follows immediately. ○

We also have the following two corresponding results. The proofs are technically similar.

THEOREM 7B. *Suppose that $I = (-\infty, B]$, where $B \in \mathbb{R}$. Suppose also that the function $f : I \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{L}([A, B])$ for every real number $A \le B$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)| \, \mathrm{d}x \le M$ for every real number $A \le B$.
Then $f \in \mathcal{L}(I)$, the limit

$$\lim_{A\to-\infty} \int_A^B f(x) \, \mathrm{d}x$$

exists, and

$$\int_{-\infty}^B f(x) \, \mathrm{d}x = \lim_{A\to-\infty} \int_A^B f(x) \, \mathrm{d}x.$$

THEOREM 7C. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{L}([A, B])$ for every $A, B \in \mathbb{R}$ satisfying $A \le B$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)| \, \mathrm{d}x \le M$ for every $A, B \in \mathbb{R}$ satisfying $A \le B$.
Then $f \in \mathcal{L}(\mathbb{R})$, the limit

$$\lim_{\substack{A\to-\infty \\ B\to\infty}} \int_A^B f(x) \, \mathrm{d}x$$

exists, and

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \lim_{\substack{A\to-\infty \\ B\to\infty}} \int_A^B f(x) \, \mathrm{d}x.$$

EXAMPLE 7.1.1. Consider the function $f:\mathbb{R}\to\mathbb{R}$, given by $f(x)=1/(1+x^2)$ for every $x\in\mathbb{R}$. It is easy to check that for every $A,B\in\mathbb{R}$ satisfying $A\le B$, we have

$$\int_A^B |f(x)|\,\mathrm{d}x=\int_A^B f(x)\,\mathrm{d}x=\tan^{-1}B-\tan^{-1}A\le\pi.$$

It follows from Theorem 7C that $f\in\mathcal{L}(\mathbb{R})$, and that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x=\lim_{\substack{A\to-\infty\\ B\to\infty}}\int_A^B f(x)\,\mathrm{d}x=\pi.$$

EXAMPLE 7.1.2. We shall demonstrate the importance of condition (b) in Theorem 7A. Define the function $f:[0,\infty)\to\mathbb{R}$ as follows: For every $n\in\mathbb{N}$, we write $f(x)=n^{-1}\sin\pi x$ for every $x\in[n-1,n)$. It is easy to check that for every real number $B\ge0$, we have

$$\int_0^B f(x)\,\mathrm{d}x=\int_0^{[B]} f(x)\,\mathrm{d}x+\int_{[B]}^B f(x)\,\mathrm{d}x,$$

where $[B]$ denotes the greatest integer not exceeding B. Then

$$\begin{aligned}\int_0^B f(x)\,\mathrm{d}x&=\sum_{n=1}^{[B]}\int_{n-1}^{n} f(x)\,\mathrm{d}x+\int_{[B]}^B f(x)\,\mathrm{d}x=\sum_{n=1}^{[B]}\int_{n-1}^{n}\frac{\sin\pi x}{n}\,\mathrm{d}x+\int_{[B]}^B\frac{\sin\pi x}{[B]+1}\,\mathrm{d}x\\&=\frac{2}{\pi}\sum_{n=1}^{[B]}\frac{(-1)^{n-1}}{n}+\frac{(-1)^{[B]}-\cos\pi B}{\pi([B]+1)},\end{aligned}$$

so that

$$\lim_{B\to\infty}\int_0^B f(x)\,\mathrm{d}x=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}+\lim_{B\to\infty}\frac{(-1)^{[B]}-\cos\pi B}{\pi([B]+1)}=\frac{2\log 2}{\pi}.$$

On the other hand, note that

$$\int_0^B |f(x)|\,\mathrm{d}x\ge\int_0^{[B]}|f(x)|\,\mathrm{d}x=\sum_{n=1}^{[B]}\int_{n-1}^{n}|f(x)|\,\mathrm{d}x=\sum_{n=1}^{[B]}\int_{n-1}^{n}\left|\frac{\sin\pi x}{n}\right|\mathrm{d}x=\frac{2}{\pi}\sum_{n=1}^{[B]}\frac{1}{n}$$

is not bounded above as $B\to\infty$, so that condition (b) fails. We shall show that $f\notin\mathcal{L}([0,\infty))$. Suppose on the contrary that $f\in\mathcal{L}([0,\infty))$. For every $N\in\mathbb{N}$, define $f_N:[0,\infty)\to\mathbb{R}$ by writing

$$f_N(x)=\begin{cases}|f(x)| & \text{if } x<N,\\ 0 & \text{if } x\ge N.\end{cases}$$

It is not difficult to see that the sequence of functions $f_N\in\mathcal{L}([0,\infty))$ is increasing on $[0,\infty)$, and that $f_N(x)\to|f(x)|$ as $N\to\infty$ for every $x\in[0,\infty)$. On the other hand, it follows from Theorem 4M that $|f|\in\mathcal{L}([0,\infty))$; also $|f_N(x)|\le|f(x)|$ for every $x\in[0,\infty)$. Hence by the Dominated convergence theorem (Theorem 6A), the sequence

$$\int_0^{\infty} f_N(x)\,\mathrm{d}x$$

is convergent. Note, however, that

$$\int_0^{\infty} f_N(x)\,\mathrm{d}x=\int_0^{N}|f(x)|\,\mathrm{d}x=\frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n}\to\infty\quad\text{as } N\to\infty,$$

a contradiction.

7.2. Improper Riemann Integrals

We now study Lebesgue integrals from the viewpoint of improper integrals.

DEFINITION. Suppose that $A \in \mathbb{R}$. Suppose further that the function $f : [A, \infty) \to \mathbb{R}$ satisfies the following conditions:
(a) $f \in \mathcal{R}([A, B])$ for every real number $B \geq A$.
(b) $\displaystyle\lim_{B\to\infty} \int_A^B f(x)\,\mathrm{d}x$ exists.
Then we say that f is improper Riemann integrable on $[A, \infty)$, and define the improper integral of f over $[A, \infty)$ by

$$\int_A^\infty f(x)\,\mathrm{d}x = \lim_{B\to\infty} \int_A^B f(x)\,\mathrm{d}x.$$

If we look at the Example 7.1.2, then we see that the existence of the improper integral does not imply the existence of the Lebesgue integral. Corresponding to Theorem 7A, we have the following result.

THEOREM 7D. *Suppose that $A \in \mathbb{R}$. Suppose further that the function $f : [A, \infty) \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{R}([A, B])$ for every real number $B \geq A$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)|\,\mathrm{d}x \leq M$ for every real number $B \geq A$.
Then both f and $|f|$ are improper Riemann integrable on $[A, \infty)$. Furthermore, $f \in \mathcal{L}([A, \infty))$, and the Lebesgue integral of f over $[A, \infty)$ is equal to the improper Riemann integral of f over $[A, \infty)$.

PROOF. Clearly

$$\int_A^B |f(x)|\,\mathrm{d}x$$

is an increasing function of B and is bounded above, so that it converges as $B \to \infty$, so that $|f|$ is improper Riemann integrable on $[A, \infty)$. On the other hand, clearly $0 \leq |f(x)| - f(x) \leq 2|f(x)|$ for every $x \in [A, \infty)$. It follows that

$$\int_A^B (|f(x)| - f(x))\,\mathrm{d}x$$

is also an increasing function of B and is bounded above, so that it also converges as $B \to \infty$. Hence

$$\int_A^B f(x)\,\mathrm{d}x$$

converges as $B \to \infty$, so that f is improper Riemann integrable on $[A, \infty)$. To complete the proof of Theorem 7D, we note that by Theorem 4V, $f \in \mathcal{L}([A, B])$ for every real number $B \geq A$, and that the Lebesgue integral of f over $[A, B]$ is equal to the Riemann integral of f over $[A, B]$. The result now follows from Theorem 7A. ◯

We also have results corresponding to Theorems 7B and 7C.