1st Hong Kong (China) Mathematical Olympiad Contest

Date: 29 December, 1998.

Time allowed: 3 hours

1. $PQRS$ is a cyclic quadrilateral with $\angle PSR = 90°$; H, K are the feet of the perpendiculars from Q to PR, PS (suitably extended if necessary) respectively. Show that HK bisects QS.

2. The base of a pyramid is a convex polygon with 9 sides. Each of the diagonals of the base and each of the edges on the lateral surface of the pyramid is coloured either black or white. Both colours are used. (Note that the sides of the base are not coloured.) Prove that there are three segments coloured the same colour which form a triangle.

3. Let s, t be given non-zero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to $(x+t,\ y-s)$. The pair (x, y) is 'good' if after some (may be, zero) number of moves it describes a pair of integers that are not relatively prime.

   (a) Determine if (s, t) is a good pair.

   (b) Show that for any s and t there is a pair (x, y) which is not good.

4. Let f be a function defined on the positive reals with the following properties:

   (1) $f(1) = 1$,

   (2) $f(x+1) = xf(x)$, and

   (3) $f(x) = 10^{g(x)}$,

   where $g(x)$ is a function defined on the reals satisfying $g[ty+(1-t)z] \le tg(y)+(1-t)g(z)$ for all y and z and for $0 \le t \le 1$.

   (a) Prove that $t[g(n)-g(n-1)] \le g(n+t)-g(n) \le t[g(n+1)-g(n)]$ where n is an integer and $0 \le t \le 1$.

   (b) Prove that $\frac{4}{3} \le f\left(\frac{1}{2}\right) \le \frac{4}{3}\sqrt{2}$.

END OF PAPER