# Practice Problem Set A for 1999 Further Training

1. Is it possible to put 100 (or 200) points on the wooden cube so that by all rotations of the cube the points map into themselves? Justify your answer.
2. Let $n$ be a positive integer. How many integer solutions $(i, j, k, l)$, $1 \le i, j, k, l \le n$, has the system of inequalities
$$\begin{cases} 1 \le -j + k + l \le n \\ 1 \le i - k + l \le n \\ 1 \le i - j + l \le n \\ 1 \le i + j - k \le n \end{cases} ?$$
3. Which fraction $\frac{p}{q}$, where $p$, $q$ are positive integers less than 100, is closest to $\sqrt{2}$? Find all digits after the point in decimal representation of that fraction which coincide with digits in decimal representation of $\sqrt{2}$ (without using calculator or table).
4. Let $a$ and $b$ be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is $q$ and the remainder is $r$. Find all pairs $(a, b)$ such that $q^2 + r = 1999$.
5. Let $n$ and $z$ be integers greater than 1 and $\gcd(n, z) = 1$. Prove
   (a) At least one of the numbers $z_i = 1 + z + z^2 + \cdots + z^i$, $i = 0, 1, \cdots, n-1$ is divisible by $n$.
   (b) If $\gcd(n, z-1) = 1$, then at least one among the numbers $z_i$, $i = 0, 1, \cdots, n-2$, is divisible by $n$.
6. Find the remainder when $2^{1990}$ is divisible by 1990.
7. For any positive integer $n$, let $s(n)$ denote the number of ordered pairs $(x, y)$ of positive integers for which $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$. Find the set of positive integers $n$ for which $s(n) = 5$.
8. For a positive integer $n$, define $A(n) = \frac{(2n)!}{(n!)^2}$. Determine the set of positive integers $n$ for which :
   (a) $A(n)$ is an even number
   (b) $A(n)$ is a multiple of 4.