WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level/Special Paper

CYD-BWYLLGOR ADDYSG CYMRU
Tystysgrif Addysg Gyffredinol
Safon Uwch/Papur Arbennig

PURE MATHEMATICS S

A.M. MONDAY, 23 June 1997

(3 hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

7086 *0023/7

1. [In this question, the lengths of the sides BC, CA and AB of the triangle ABC are denoted by a, b and c respectively.]

An acute-angled triangle ABC has $AC > AB$. The mid-point of BC is denoted by D and the foot of the perpendicular from A to BC is denoted by E.

(*a*) Show that

(i) $b^2 - c^2 = a(b\cos C - c\cos B)$,

(ii) $DE = \dfrac{b^2 - c^2}{2a}$,

(iii) $2\cot A\widehat{D}B = \cot C - \cot B$. [11]

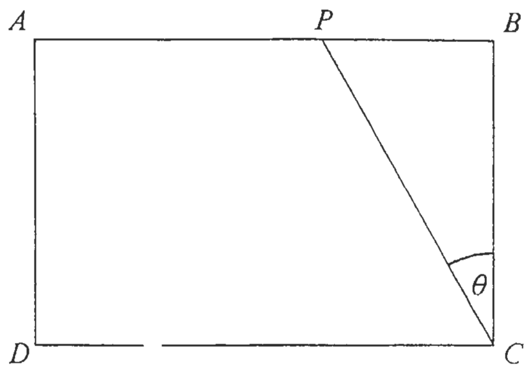
(*b*) Given that $b = 4$, $c = 3$ and $A = 80°$, find the length of AD. [6]

2. (*a*) The point $P(x,y)$ moves in such a way that its distance from the fixed point $F(ae, 0)$ is equal to e times its distance from the fixed line $x = \dfrac{a}{e}$, where $0 < e < 1$. Show that P lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2)$. [3]

(*b*) The perpendicular from the origin O to the tangent to the ellipse at $Q(a\cos\theta, b\sin\theta)$ meets the line FQ at the point R. Show that, as θ varies, R moves in a circle with centre F. [14]

3. 

The diagram shows a rectangular ploughed field $ABCD$ with $AB = a$ metres and $BC = b$ metres, where $a > b$. There is a footpath along the edges AB and BC of the field. A runner, arriving at A, wishes to run to C as quickly as possible. He can run along the footpath with speed u m s^{-1} and across the field with speed v m s^{-1}, where $v < u$. He decides to run along the footpath to a point P between A and B, and then to run across the field from P to C. The angle $B\widehat{C}P$ is denoted by θ.

Write down an expression for the total time, T s, taken to go from A to C in terms of a, b, u, v and θ. [2]

(*a*) (i) Show that T has a minimum value when $\sin\theta = \dfrac{v}{u}$.

(ii) Show that the minimum value of T can be written in the form

$$\frac{a}{u} + \frac{b\sqrt{u^2 - v^2}}{uv}.$$ [9]

(*b*) Show that the route corresponding to this value of θ gives the quickest route from A to C only if

$$\frac{v}{u} > \frac{1}{\sqrt{2}} \quad \textbf{and} \quad \frac{v}{u} < \frac{a}{\sqrt{a^2 + b^2}}.$$ [6]

4. The function f with domain, $[0, \infty)$ is defined by

$$f(x) = e^{-ax} \mid \sin bx \mid$$

where a, b are positive constants.

(*a*) Sketch the graph of f. [2]

(*b*) Use integration by parts to show that

$$\int_0^{\frac{\pi}{b}} f(x)\,\mathrm{d}x = \frac{b(1 + e^{-\frac{a\pi}{b}})}{a^2 + b^2}.$$ [7]

(*c*) Use the substitution $y = x - \frac{\pi}{b}$ to evaluate

$$\int_{\frac{\pi}{b}}^{\frac{2\pi}{b}} f(x)\,\mathrm{d}x.$$ [4]

(*d*) Find an expression for the total area enclosed between the graph of f and the x-axis. [4]

5. (*a*) By writing 4^n as $(1 + 3)^n$, show that $4^n - 1$ is divisible by 3 for all positive integral values of n. Hence, using mathematical induction, prove that $4^n - 3n + 8$ is divisible by 9 for all positive integral values of n. [5]

(*b*) Use mathematical induction to prove that

$$\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}^n = n.2^{n-1}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + 2^n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for all positive integral n. [6]

(*c*) Given that

$$u_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$$

where n is a positive integer, show that

$$u_{n+2} - 6u_{n+1} + 4u_n = 0.$$

Use mathematical induction to prove that u_n is an integer for all n. [6]

6. The matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

(*a*) Find the values of k for which the determinant of the matrix $(\mathbf{A} - k\mathbf{I})$ is equal to zero, where **I** denotes the 2×2 identity matrix. [3]

(*b*) For each value k_i $(i = 1, 2)$, find a 2×1 matrix $\mathbf{X}_i$ which satisfies the equations

$$\mathbf{AX}_i = k_i\,\mathbf{X}_i \text{ and } \mathbf{X}_i'\,\mathbf{X}_i = 1$$

where $\mathbf{X}_i'$ denotes the transpose of $\mathbf{X}_i$. [6]

(*c*) **H** is the 2×2 matrix whose ith column $(i = 1, 2)$ is $\mathbf{X}_i$. Verify that

$$\mathbf{A} = \mathbf{HKH}^{-1}$$

$$\text{where } \mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$ [4]

(*d*) The 2×2 matrix **B** is called a square root of **A** if $\mathbf{B}^2 = \mathbf{A}$.

Show that

$$\mathbf{H}\begin{bmatrix} \sqrt{k_1} & 0 \\ 0 & \sqrt{k_2} \end{bmatrix}\mathbf{H}^{-1}$$

is a square root of **A**.

Evaluate this square root. [4]

7. The complex numbers $z = x + iy$ and $w = u + iv$, where x, y, u and v are real, are related by

$$w = f(z) = \frac{2z+1}{z-2}.$$

(*a*) Show that

$$u = \frac{2x^2 + 2y^2 - 3x - 2}{x^2 + y^2 - 4x + 4}$$

and obtain an expression for v in terms of x and y. [4]

(*b*) (i) Verify that

$$ff(z) = z.$$

(ii) Hence, or otherwise, obtain expressions for x and y in terms of u and v. [5]

(*c*) The relationship between z and w defines a transformation T in which the point (x, y) in the z-plane is transformed to the point (u, v) in the w-plane.

(i) Show that, under T, the line $y = x + 1$ is transformed into a circle. State its radius and the coordinates of its centre.

(ii) Find and identify an equation for the image under T of the circle $|z - 1| = 1$. [8]

8. The curve C has equation

$$y = -\ln(1 - x^2) \text{ for } 0 \leqslant x < 1.$$

The point P, whose x-coordinate is a, lies on C.

(*a*) (i) Show that the arc length of C from the origin O to the point P is equal to

$$\ln(1 + a) - \ln(1 - a) - a.$$ [6]

(ii) Use the Newton-Raphson method to find, correct to four decimal places, the value of a for which this arc length is equal to 1. [6]

(*b*) The point $Q\left(\frac{1}{\sqrt{2}}, \ln 2\right)$ lies on C. The region enclosed between C, the y-axis and the abscissa through Q is rotated through four right-angles about the y-axis. Show that the curved surface area of the solid generated is given by

$$\pi \int_0^{\ln 2} (2 - e^{-y})\, dy.$$

[You are not required to evaluate this integral.] [5]