WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level/Special Paper

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch/Papur Arbennig

MATHEMATICS S

A.M. FRIDAY, 26 June 1998

(3 hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

The part of the syllabus to which each question is addressed is indicated after the question number.
e.g. Question 1 (A1, P1)
This question is suitable for candidates who have pursued the Mathematics A1 or Mathematics (Modular) P1.

The only books of statistical tables that you may use in the examination are "Statistical Tables" by Murdoch and Barnes (Macmillan Press) **or** "Elementary Statistical Tables" (RND Publications).
Take $g = 9{\cdot}8 \text{ ms}^{-2}$.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

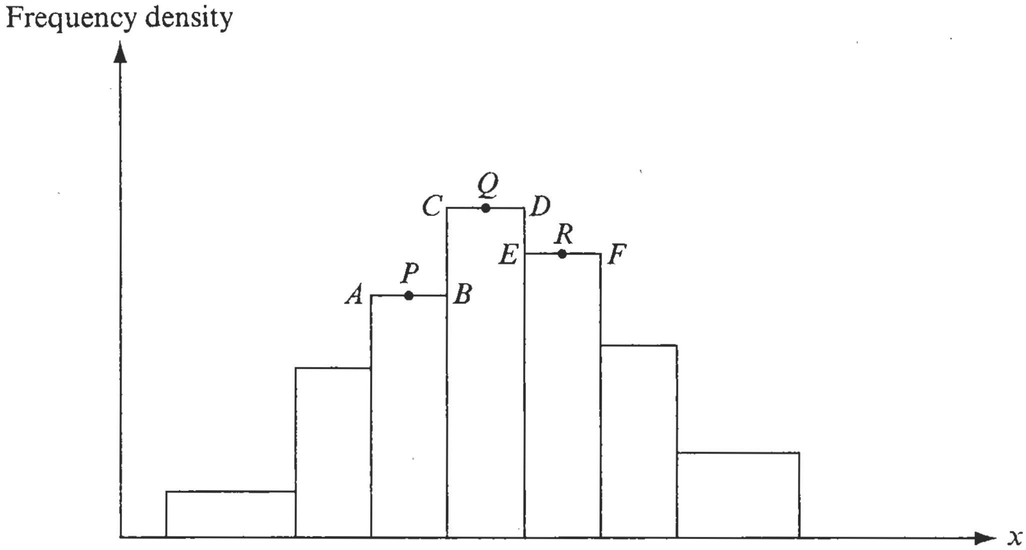
The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

SECTION A

Mathematical Methods

Question 1. (A1, P1/P2)



The diagram shows a histogram illustrating a grouped frequency distribution. The three central rectangles of this histogram have heights h_1, h_2 and h_3 and each has unit width. The mid-points of the sides AB, CD and EF of these rectangles are $P(q-1,\ h_1)$, $Q(q,\ h_2)$ and $R(q+1,\ h_3)$ respectively. A curve with equation of the form $y = a + bx + cx^2$ passes through P, Q and R. Determine expressions for b and c in terms of h_1, h_2, h_3 and q. The mode of the distribution is estimated by the x-coordinate of the maximum point on this curve.

(*a*) Show that this estimate is given by

$$q + \frac{(h_3 - h_1)}{2(2h_2 - h_1 - h_3)}.$$ [11]

(*b*) Show that this estimate is the x-coordinate of the point of intersection of BD and CE. [6]

Question 2. (A1, P1/P2)

The equations of two circles are respectively

$$x^2 + y^2 - 2x - 4y - 20 = 0$$
$$\text{and } x^2 + y^2 - 10x - 12y + \lambda = 0.$$

(*a*) Given that the circles intersect in two distinct points, show that

$$4 - 40\sqrt{2} < \lambda < 4 + 40\sqrt{2}.$$ [7]

(*b*) Given that $\lambda = 60$, find the area of the region that lies inside both circles. [10]

Question 3. (A1, P1/P2)

The function f is defined, for all x, by

$$f(x) = \sin^m x \sin mx + \cos^m x \cos mx$$

where m is a positive integer greater than 1.

Show that f has a stationary value where $x = \dfrac{\pi}{4}$. [5]

Identify the type of stationary value when

(*a*) m is a multiple of 8, [5]

(*b*) m is an odd multiple of 4, [3]

(*c*) $m = 3$. [4]

Question 4. (A1, P1/P2)

A bank lends £P to a borrower at the beginning of a particular month at an interest rate of I% per month. Interest is calculated at the end of each month and added to the amount outstanding. The borrower repays a fixed amount £R at the end of each month, after the interest has been added. Given that £S_n denotes the amount still outstanding immediately after the nth monthly payment has been made, show that

$$S_n = P\left(1+\frac{I}{100}\right)^n - \frac{100R}{I}\left[\left(1+\frac{I}{100}\right)^n - 1\right].$$ [6]

(*a*) Huw borrows £50 000 at a monthly interest rate of 0·7%, arranging to repay £450 per month. How long does it take for the loan to be repaid completely? [6]

(*b*) Megan borrows £30 000 and the bank informs her that a monthly repayment of £400 will repay the loan in 10 years. Find, correct to two decimal places, the monthly interest rate being charged. [5]

SECTION B

Mechanics

Question 5. (A2, A5, M1)

The point Q is at a height h above a smooth horizontal plane. A second point P is directly above Q and $PQ = h$. Small smooth spheres A and B are dropped from rest simultaneously from P and Q respectively. The spheres collide before sphere B hits the plane for the second time. The coefficient of restitution between sphere B and the plane is e.

(*a*) Show that the time that elapses between sphere B hitting the plane for the first time and the spheres colliding is $\frac{1}{1+e}\sqrt{\frac{h}{2g}}$. [7]

(*b*) Find the height above the plane at which the spheres first collide. [2]

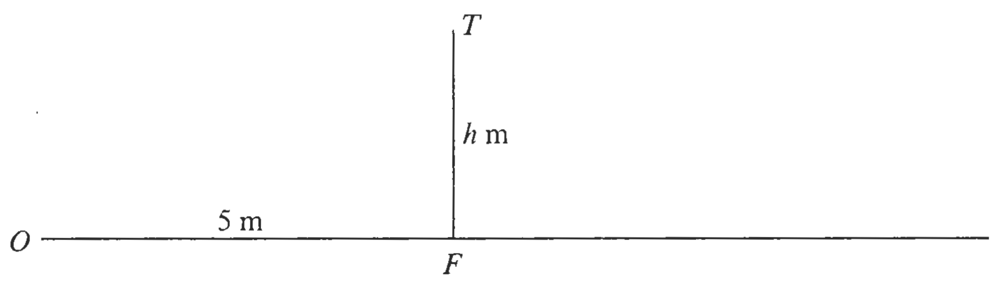
(*c*) Show that $e > \frac{\sqrt{2}-1}{2}$. [5]

(*d*) Find, as e varies, the maximum height above the plane of the point of collision. [3]

Question 6. (A2, A5, M1)

(In this question you may assume that the range on a horizontal plane of a particle projected with speed u at an angle α above the horizontal is $\frac{2u^2 \sin\alpha\cos\alpha}{g}$.)

The speed, V ms^{-1}, of projection of a ball is such that the maximum range that can be attained on a horizontal plane is 12·5 m. Find V^2 in terms of g. [3]



The diagram shows a point O on horizontal ground and a vertical wall FT of height h m. The point O is 5 m from the foot F of the wall. A ball is projected from O with the above speed V ms^{-1} at an angle of elevation α so that it goes over the wall. The plane of motion of the ball is perpendicular to the wall. As α varies, the nearest and furthest points of landing of the ball on the ground beyond F from the wall are A and B respectively.

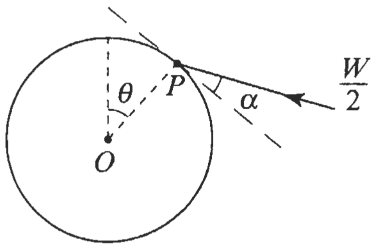
(*a*) For the case $h = 5$ m, show that

$$\tan^2\alpha - 5\tan\alpha + 6 \leqslant 0,$$

and find the length of AB. [8]

(*b*) Find the length of AB for the case $h = 1{\cdot}25$ m. [6]

Question 7. (A2, M2)



The diagram shows a rough vertical hoop with centre O and a bead of weight W, threaded on the hoop at P, where OP is inclined at an angle θ to the upward vertical. The bead is prevented from sliding down the hoop by a force of magnitude $\frac{W}{2}$ acting in the direction at an angle α to the tangent to the hoop at P, as shown.

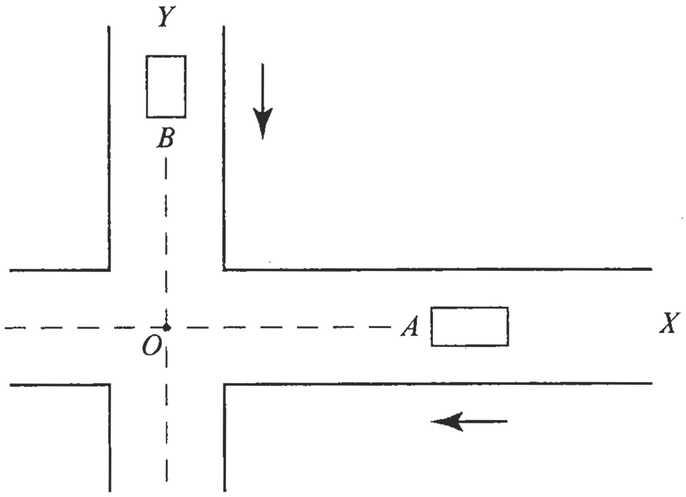
The coefficient of friction between the bead and the wire is $\tan\left(\frac{\pi}{6}\right)$.

(*a*) Show that

$$\sin\left(\theta - \frac{\pi}{6}\right) \leqslant \frac{1}{2}\cos\left(\alpha - \frac{\pi}{6}\right).$$ [12]

(*b*) Find the maximum value of θ and the corresponding value of α. [5]

Question 8 (A2, M2)



The diagram shows two perpendicular tracks OX and OY. A toy car is travelling towards O along the track OX with constant speed V_A and the midpoint of the front of this car is denoted by A. Another toy car is travelling towards O along the track OY with constant speed V_B and the midpoint of its front is denoted by B. Unit vectors in the directions OX and OY are denoted by **i** and **j** respectively and, at time $t = 0$, $OA = a$ and $OB = b$.

(*a*) Modelling the cars as particles, write down, in terms of **i**, **j** t, a, b, V_A and V_B, the vector ***AB*** at time t and obtain an expression, which need not be simplified, for AB^2. [4]

(*b*) The cars are now to be modelled as thin rods of length L. The point P is on the car on OX at a distance x from A and the point Q is on the other car at a distance of y from B.

(i) Write down an expression for the vector ***PQ*** at time t. [3]

(ii) Find the ratio $\frac{V_B}{V_A}$ such that the front of the first car strikes the second car at the point Q and show that, for the front of the first car to strike the side of the second car,

$$\frac{b}{a} \leqslant \frac{V_B}{V_A} \leqslant \frac{b+L}{a}.$$ [5]

(iii) Write down the inequality that would have to be satisfied for the front of the second car to strike the side of the first car. [3]

(iv) Find the range of values $\frac{V_B}{V_A}$ so that the cars collide. [2]

SECTION C

Probability and Statistics

Question 9. (A3, A5, S1)

A sales representative uses his car to visit his customers only on the Monday, Tuesday, Wednesday and Thursday of each week. The distance he travels during any particular **week** is independent of the distance he travels during any other week and may be modelled by the normal distribution having mean 440 miles and standard deviation 60 miles.

Under the terms of a new contract he will be paid his travelling expenses at the end of that week in which his **total** mileage reaches 2000 miles.

(*a*) Show that the probability that he will not have been paid his travelling expenses by the end of the fourth week of his new contract is 0·977, correct to three decimal places. [4]

(*b*) Find the probability that he will be paid his travelling expenses at the end of the fifth week of his new contract. Give your answer correct to three decimal places. [6]

(*c*) It may also be assumed that the distance the sales representative travels on any particular **day** is independent of the distance he travels on any other day and may be modelled by the normal distribution having mean μ miles and standard deviation σ miles.

Given that the sales representative has not been paid his travelling expenses by the end of the fourth week of his new contract, find the probability that he will still not have covered a total of 2000 miles by the following Monday night.

Give your answer correct to two decimal places. [7]

Question 10. (A3, A5, S1)

(*a*) Given that

$$1 + q + q^2 + q^3 + \ldots = \frac{1}{(1-q)},$$

show, by differentiating both sides, that

$$1 + 2q + 3q^2 + 4q^3 + \ldots = \frac{1}{(1-q)^2},$$

and deduce that

$$2 + 6q + 12q^2 + 20q^3 + \ldots = \frac{2}{(1-q)^3}.$$ [2]

John has to prepare for an examination in Geography. In order to help him revise his work, John's Geography teacher decides to ask him a series of multiple choice questions. Each question has n possible answers, only one of which is correct. John knows very little about Geography, and therefore decides to guess at random the correct answer to each question in turn. Let X denote the number of the first question which John answers correctly.

(*b*) Write down an expression, in terms of r and n, for $P(X = r)$, $r = 1, 2, 3, \ldots$, and verify that the sum of all these probabilities equals 1. [3]

(*c*) Find expressions in terms of n for

(i) $E(X)$,

(ii) $E[X(X-1)]$,

(iii) $\text{Var}(X)$. [8]

(*d*) Let Y denote the number of the second question which John answers correctly. Find $E(Y)$. [4]

Question 11. (A3, S2)

(*a*) The random variable X has a Poisson distribution with unknown mean μ. The mean of a random sample of n observations on X is denoted by $\overline{X}$. Using a normal approximation for the distribution of $\overline{X}$, show that

$$P\left[\frac{n(\overline{X}-\mu)^2}{\mu} \leqslant 3\cdot 8416\right] \approx 0\cdot 95.$$

By treating the above inequality as a quadratic in μ, show that approximate 95% confidence limits for μ are given by

$$\frac{2n\overline{x}+3\cdot 8416 \pm 1\cdot 96\sqrt{4n\overline{x}+3\cdot 8416}}{2n}$$

where $\overline{x}$ is the observed value of $\overline{X}$. [9]

(*b*) The numbers of eggs hatched during the breeding season by female birds of a particular species are modelled by a Poisson distribution. In order to estimate the population mean, μ, a researcher monitors 20 such birds and the results obtained are summarised in the following table.

Number of eggs hatched	0	1	2	3	4	5
Frequency	1	3	5	7	3	1

(i) Use the result given in (*a*) to find an approximate 95% confidence interval for μ.

(ii) A schoolgirl notices that 3 female birds of the species are nesting in her garden. Calculate an approximate 95% confidence interval for the probability that these 3 birds will hatch a total of no more than 4 eggs. [8]

Question 12. (A3, S2)

A car salesman sold six cars of the same model in a particular week. The selling prices and ages of these cars were as follows.

Age (x months)	12	26	8	30	16	2
Selling price (£y)	6300	4850	7550	4050	6100	8350

It is decided to fit the model

$$y = \lambda e^{-\mu x}$$

to these data, where λ and μ are constants.

(*a*) Show that ln y and x are linearly related. [1]

(*b*) Hence find the least squares estimates of λ and μ. [7]

(*c*) The measurements of x are exact whereas the values of ln y may be assumed to be subject to independent normally distributed random errors with mean zero and standard deviation 0·05.

(i) Find a 95% confidence interval for λ.

(ii) Find and interpret the p-value of your estimated value of μ with respect to the hypotheses

$$H_0: \mu = 0{\cdot}03 \text{ versus } H_1: \mu \neq 0{\cdot}03$$ [9]