K

WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level/Special Paper

CYD-BWYLLGOR ADDYSG CYMRU
Tystysgrif Addysg Gyffredinol
Safon Uwch/Papur Arbennig

PURE MATHEMATICS S

A.M. FRIDAY, 26 June 1998

(3 Hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

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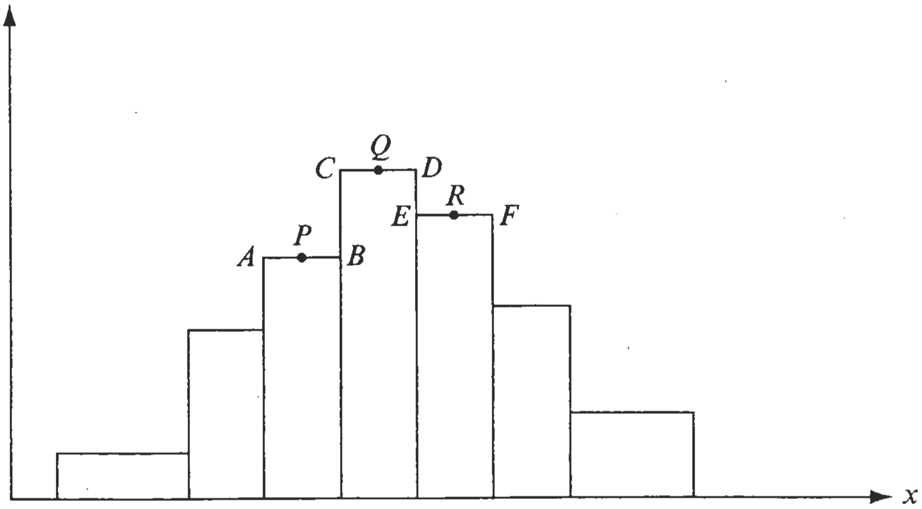
1. Frequency density



The diagram shows a histogram illustrating a grouped frequency distribution. The three central rectangles of this histogram have heights h_1, h_2 and h_3 and each has unit width. The mid-points of the sides AB, CD and EF of these rectangles are P $(q-1, h_1)$, Q (q, h_2) and R $(q+1, h_3)$ respectively. A curve with equation of the form $y = a + bx + cx^2$ passes through P, Q and R. Determine expressions for b and c in terms of h_1, h_2, h_3 and q. The mode of the distribution is estimated by the x-coordinate of the maximum point on this curve.

(*a*) Show that this estimate is given by

$$q + \frac{(h_3 - h_1)}{2(2h_2 - h_1 - h_3)}.$$ [11]

(*b*) Show that this estimate is the x-coordinate of the point of intersection of BD and CE. [6]

2. The equations of two circles are respectively

$$x^2 + y^2 - 2x - 4y - 20 = 0$$
$$\text{and } x^2 + y^2 - 10x - 12y + \lambda = 0.$$

(*a*) Given that the circles intersect in two distinct points, show that

$$4 - 40\sqrt{2} < \lambda < 4 + 40\sqrt{2}.$$ [7]

(*b*) Given that $\lambda = 60$, find the area of the region that lies inside both circles. [10]

3. The function f is defined, for all x, by

$$f(x) = \sin^m x \sin mx + \cos^m x \cos mx$$

where m is a positive integer greater than 1.

Show that f has a stationary value where $x = \frac{\pi}{4}$. [5]

Identify the type of stationary value when

(*a*) m is a multiple of 8, [5]

(*b*) m is an odd multiple of 4, [3]

(*c*) $m = 3$. [4]

4. A bank lends £P to a borrower at the beginning of a particular month at an interest rate of I% per month. Interest is calculated at the end of each month and added to the amount outstanding. The borrower repays a fixed amount £R at the end of each month, after the interest has been added. Given that £S_n denotes the amount still outstanding immediately after the nth monthly payment has been made, show that

$$S_n = P\left(1+\frac{I}{100}\right)^n - \frac{100R}{I}\left[\left(1+\frac{I}{100}\right)^n - 1\right].$$ [6]

(*a*) Huw borrows £50 000 at a monthly interest rate of 0·7%, arranging to repay £450 per month. How long does it take for the loan to be repaid completely? [6]

(*b*) Megan borrows £30 000 and the bank informs her that a monthly repayment of £400 will repay the loan in 10 years. Find, correct to two decimal places, the monthly interest rate being charged. [5]

5. (*a*) Show that the transformation of the plane in which each point is transformed to its reflection in the line $y = x\tan\alpha + c$ is represented by the matrix

$$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha & -c\sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha & c(1+\cos 2\alpha) \\ 0 & 0 & 1 \end{bmatrix}.$$ [5]

(*b*) The ellipse C_1 has equation

$$3x^2 + y^2 = 1$$

and C_2 is the reflection of C_1 in the tangent to C_1 at the point $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$. Show that the equation of C_2 is

$$3x^2 + 5y^2 + 2\sqrt{3}xy - 4\sqrt{6}x - 8\sqrt{2}y + 8 = 0.$$ [12]

6. (*a*) Starting with

$$e^{i\theta} = \cos\theta + i\sin\theta$$

show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and find a similar expression for $\sin\theta$. Hence show that

$$\cos i\theta = \cosh\theta$$
$$\text{and } \sin i\theta = i\sinh\theta.$$ [5]

(*b*) Find, in Cartesian form, all complex numbers z satisfying

$$e^z = -2.$$ [6]

(*c*) Find, in Cartesian form, all complex numbers z satisfying

$$\sin z = 3.$$ [6]

7. (*a*) Given that the curves $y = e^{3x}$ and $y = k\sqrt{x}$ touch each other, show that

$$k = \sqrt{6e}.$$ [6]

(*b*) By sketching suitable graphs, or otherwise, find the range of values of k for which the equation

$$e^{3x} - k\sqrt{x} = 0$$

has two distinct real roots. [3]

(*c*) Find, correct to four decimal places, both real roots of the equation

$$e^{3x} - 4{\cdot}04\sqrt{x} = 0.$$ [8]

8. Given that

$$I_n = \int_0^1 (x^2 - x + 1)^n \, dx$$

show that

$$(2n+1)I_n = 1 + \frac{3n}{2} I_{n-1}.$$ [8]

Use the above reduction formula to evaluate I_n when

(*a*) $n = \frac{3}{2}$, [7]

(*b*) $n = -\frac{3}{2}$ [2]