WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level/Special Paper

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025/09

APPLIED MATHEMATICS S

A.M. FRIDAY, 30 June 2000

(3 hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

The only books of statistical tables that you may use in the examination are "Statistical Tables" by Murdoch and Barnes (Macmillan Press) **or** "Elementary Statistical Tables" (RND Publications).

Take $g = 9{\cdot}8\ \text{ms}^{-2}$.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

1. A particle is projected from a point A on the horizontal ground with velocity u ms^{-1} at an angle α above the horizontal. The particle just clears the top of a vertical wall of height h m. The foot of the wall is on the horizontal ground, d m from A.

(*a*) Show that $\frac{gd^2}{2u^2}\tan^2\alpha - d\tan\alpha + h + \frac{gd^2}{2u^2} = 0.$ [6]

(*b*) Show that $u^4 - 2ghu^2 - g^2d^2 \geqslant 0.$ [3]

(*c*) Show that the minimum possible value of u is $\sqrt{g\left[h + (d^2 + h^2)^{\frac{1}{2}}\right]}$ [5]

(*d*) Given that $d = 40$ and $h = 30$, find the value of α corresponding to this minimum value of u. [3]

2. (*a*) Two balls B_1 and B_2, of equal radii but of mass m_1 and m_2 respectively, are moving on a smooth horizontal table directly towards each other with speeds u_1 and u_2 respectively. The coefficient of restitution between the balls is e. The ball B_1 is brought to rest by the collision. Find, in terms of e, u_1 and u_2, the speed of B_2 after the collision, and the ratio $m_1 : m_2$. [6]

(*b*) Two balls P and Q, of equal radii, are moving in the same direction with P ahead of Q, on a smooth horizontal floor along a straight line which is perpendicular to a smooth wall. The speeds of P and Q are $4U$ and U respectively. First the ball P, which is of mass M, hits the wall, rebounds, collides with Q and is brought to rest by this second collision. The coefficient of restitution between P and the wall is $\frac{1}{2}$ and the coefficient of restitution between the two balls is e. Find, in terms of e and U, the speed of Q after the second collision and determine the mass of Q in terms of e and M. [6]

The ball Q now collides directly with another ball R with the same radius, moving towards it with speed U. In this collision Q is brought to rest. The coefficient of restitution between these two balls is also e. Find, in terms of e and U, the speed of R after this collision and determine the mass of R in terms of e and M. [5]

3. At time t s, a car of mass m kg is travelling at a variable speed of v ms^{-1} on a horizontal road. The resistance to the motion of the car is modelled by a horizontal force of magnitude kv^2 N, where k is a positive constant. The car's engine is working at a constant rate of P W and the maximum speed that the car can sustain on the road when the engine is working at this rate is u ms^{-1}. Show that $k = \dfrac{P}{u^3}$. [3]

At the point A, the speed of the car is $\dfrac{2u}{5}$ ms^{-1} and, at the point B, the speed of the car is $\dfrac{3u}{5}$ ms^{-1}. Show that the distance AB is $\dfrac{mu^3}{3P}\ln\left(\dfrac{117}{98}\right)$ metres. [7]

The engine is switched off as the car passes through B. The car then decelerates under the action of the resistance and again attains a speed of $\dfrac{2u}{5}$ ms^{-1} at the point C. Find the ratio of the distance BC to the distance AB in the form $n : 1$. Give the value of n correct to three significant figures. [7]

4. (*a*) A small bead of mass m is threaded on a smooth circular hoop of radius $2a$, which is fixed in a vertical plane. One end of a light elastic string of natural length a and modulus $\dfrac{3mg}{4}$ is tied to the bead and the other end is tied to A, the highest point of the hoop. The bead is slightly displaced from rest at B, the lowest point of the hoop. It may be assumed that the string remains taut throughout the motion.

(i) Show that the elastic energy stored in the string is given by $\dfrac{3mga(4\cos\theta - 1)^2}{8}$, where θ denotes the angle between the string and the vertical.

(ii) Find the value of $\cos\theta$ when the bead is at its greatest height above B in the subsequent motion.

(iii) Find the value of $\cos\theta$ when the bead is travelling at its greatest speed. [15]

(*b*) The string is now replaced by another string of natural length a but whose modulus is greater than $\dfrac{3mg}{4}$. Without any further calculation, explain carefully why it may not now be possible to answer (*a*)(ii) without modifying your mathematical model. [2]

5. The working day in a particular factory consists of two shifts; the morning shift and the afternoon shift. The number of accidents that occur during a morning shift has a Poisson distribution with mean $1{\cdot}2$, and, independently, the number of accidents that occur during an afternoon shift has a Poisson distribution with mean $1{\cdot}8$.

(*a*) Given that 5 accidents occurred during a particular working day, find the conditional probability that 2 of these accidents occurred during the morning shift. [4]

(*b*) Given that at least 1 accident occurred during a particular working day, find the conditional probability that all of the accidents which occurred that particular day occurred during the same shift. Give your answer correct to three decimal places. [4]

(*c*) The proportion of the accidents occurring in this factory which are classified as being serious accidents is 30%. Find, correct to three decimal places, the probability that on a particular working day,

(i) 4 accidents will occur, 2 of which are serious accidents,

(ii) no serious accident will occur. [9]

6. A company manufactures two types of electrical components which are identical in appearance. The lifetimes of components of type A are normally distributed with mean 160 hours and standard deviation 8 hours. The lifetimes of components of type B are normally distributed with mean 150 hours and standard deviation 6 hours.

(*a*) A particular dealer buys a large batch of these components. This batch contains twice as many components of type A than of type B. Two components are chosen at random from this batch. Find the probability that the greater of the lifetimes of the two chosen components will be at least 156 hours. [6]

(*b*) Another dealer buys two large batches of components from the company. One batch contains components of type A only and the other batch contains components of type B only, but the dealer does not know which batch is which.

(i) In order to try to identify which batch contains components of type A and which batch contains components of type B, the dealer chooses at random one component from each batch. Both components are then tested and the dealer decides to assume that the batch which originally contained the component with the longer lifetime is the batch containing components of type A. Show that there is slightly less than one chance in six that the dealer's assumption will be incorrect.

(ii) The dealer now refines the above process by choosing at random n components from each batch. All $2n$ components are then tested and the dealer decides to assume that the batch which originally contained those n components with the greater total lifetime is the batch containing components of type A. Find the least value of n so that there is less than one chance in a thousand that the dealer's assumption will be incorrect. [11]

7. A random sample of 100 observations on the discrete random variable X yielded the following frequency table.

x	0	1	2	3	4	5 or more
Frequency	52	20	15	9	4	0

(*a*) Find an approximate 95% confidence interval for

(i) μ where $\mu = E(X)$,

(ii) θ where $\theta = P(X = 0 \text{ or } 1)$. [8]

The probability distribution of X is given by

$$P(X = x) = \alpha^x(1 - \alpha) \quad \text{for } x = 0, 1, 2, \ldots$$

where α is an unknown parameter between 0 and 1.

(*b*) Given that $\mu = \dfrac{\alpha}{(1-\alpha)}$,

(i) deduce an approximate 95% confidence interval for α,

(ii) find, approximately, the p-value of the above data with respect to the hypotheses

$$H_0 : \alpha = 0{\cdot}4 \quad \text{versus} \quad H_1 : \alpha \neq 0{\cdot}4.$$ [7]

(*c*) Find another approximate 95% confidence interval for α using the result obtained in (*a*)(ii). [2]

TURN OVER

8. The relationship between the variables x, y is given by $y = \lambda x$, where λ is an unknown constant. To estimate λ, n pairs of measurements

$$(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$$

are made on x and y. The x measurements are exact, but the y measurements are subject to independent normally distributed errors with mean zero and standard deviation σ.

(*a*) Two possible estimates of λ are given by

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$\text{and } \hat{\lambda}_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i}.$$

(i) Show that both estimates are unbiased.

(ii) Show that

$$\text{Var}(\hat{\lambda}_1) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

$$\text{and } \text{Var}(\hat{\lambda}_2) = \frac{\sigma^2}{n^2}\sum_{i=1}^{n} \frac{1}{x_i^2}.$$

[10]

(*b*) The following measurements were made on x and y.

x	1	2	3	4	5
y	1·72	3·50	5·56	7·23	9·11

Determine which of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ is the better estimate, and calculate its value correct to two decimal places. [7]