WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level/Special Paper

CYD-BWYLLGOR ADDYSG CYMRU
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Safon Uwch/Papur Arbennig

024/09

PURE MATHEMATICS S

A.M. FRIDAY, 30 June 2000

(3 Hours)

INSTRUCTIONS TO CANDIDATES

Answer **six** questions.

INFORMATION FOR CANDIDATES

An electronic calculator will be required.

The booklet "Information for the use of candidates in Mathematics" is available and may be used.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

1. The equations of the three sides of the triangle T are respectively

$$x - 2y + 5 = 0,\ x - 7y + 30 = 0,\ 3x - y + 10 = 0.$$

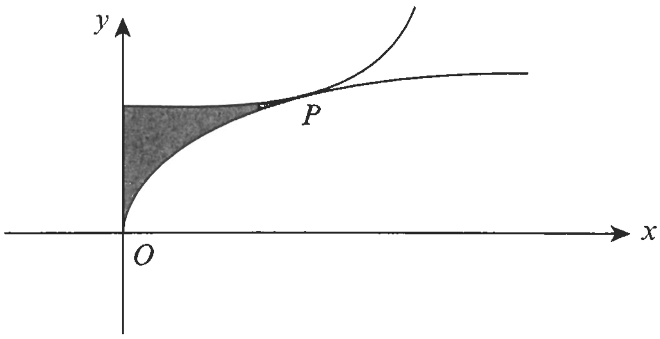
(*a*) Find the area of T. [9]

(*b*) (i) Find the equation of the circle that passes through the three vertices of T.

(ii) The point P is on the positive x-axis and the two tangents from P to this circle are perpendicular. Find the coordinates of P. [8]

2. (*a*) The function f is a polynomial and f' denotes its derivative. Given that α is a multiple root of the equation $f(x) = 0$, show that α is also a root of the equation $f'(x) = 0$. [4]

(*b*)

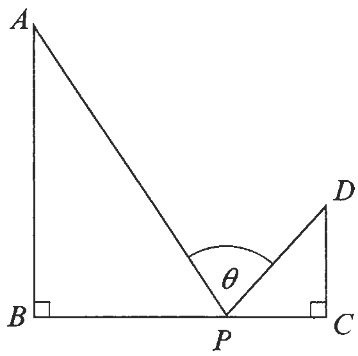


The figure shows parts of the curves $y^2 = x$ and $y = x^2 + k$ touching at the point P.

(i) Find the coordinates of P and the value of the constant k.

(ii) Find the area of the shaded region in the form $\frac{1}{n}$, where n is a positive integer. [13]

3.

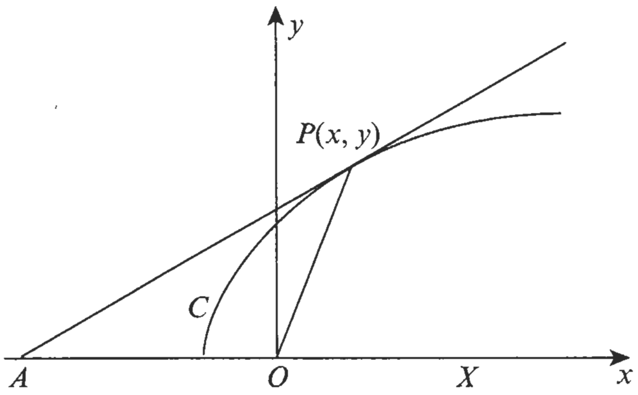


In the diagram above, AB and CD are both perpendicular to the line BC; $AB = 60$ units, $BC = 50$ units and $CD = 30$ units; P denotes a point that lies on BC between B and C, $BP = x$ units and $A\widehat{P}D = \theta°$.

(*a*) Find the greatest and least values of θ, in degrees correct to two decimal places, as P moves from B to C. [10]

(*b*) Find, in the form $[M, N]$ with M, N correct to the nearest integer, the range of values of x for which $\theta \geqslant 60°$. [7]

4.



(*a*) The above figure shows the curve C with equation $y^2 = 4ax + 4a^2$. $P(x, y)$ is a general point on C. The tangent to C at P meets the x-axis at A. Show that

$$\tan P\widehat{A}O = \frac{2a}{y}$$

and write down an expression for $\tan P\widehat{O}X$ in terms of x and y. Using an appropriate double angle formula, show that

$$P\widehat{O}X = 2P\widehat{A}O.$$ [5]

(*b*) Conversely, suppose now that the equation of C is not known but you are given that $P\widehat{O}X = 2P\widehat{A}O$ for all positions of P. Show that

$$2x = \frac{y}{p} - py, \qquad \text{where } p = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

By first differentiating this equation with respect to y, show that C has an equation of the form $y^2 = Dx + E$, where D and E are constants. [12]

5. Three transformations in the plane in which the point (x, y) is transformed to the point (x', y') are given by

$$T_1: \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 3 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$T_2: \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0{\cdot}6 & -0{\cdot}8 & 2 \\ -0{\cdot}8 & -0{\cdot}6 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$T_3: \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

One of these transformations is a rotation, one is a reflection and one is neither a rotation nor a reflection.

Determine which is which.

(i) For the rotation, find the angle of rotation and the coordinates of the centre.

(ii) For the reflection, find the equation of the line of reflection.

(iii) For the remaining transformation, show that it is an isometry. [17]

6. Show that

$$\int_2^3 \frac{dx}{x^3 - 1} = \frac{1}{6} \ln\left(\frac{28}{13}\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{19}\right).$$ [17]

7. (*a*) By drawing suitable graphs, determine the number of real roots of the equation

$$x = e^x.$$ [2]

(*b*) The equation

$$z = e^z \qquad \text{(I)}$$

has a complex root denoted by $a + ib$.

(i) Show that

$$\frac{b}{\sin b} = e^{\frac{b}{\tan b}}$$

and prove that this equation has a root between 1·3 and 1·35.

(ii) Verify that the equation in (i) can be rewritten in the form

$$b = \tan^{-1}\left[\frac{b}{\ln\left(\frac{b}{\sin b}\right)}\right].$$

Use an iterative method based on this rearrangement to find the value of the root correct to four decimal places.

(iii) Hence find a complex root of equation (I), giving the real and imaginary parts correct to three decimal places. [15]

8. (*a*) Given that

$$I_n = \int_0^{\pi} \left(\frac{1 - \cos n\theta}{1 - \cos \theta}\right) d\theta$$

where n is a positive integer, show that

$$I_n + I_{n+2} - 2I_{n+1} = 0.$$ [5]

(*b*) (i) Evaluate I_1 and I_2.

(ii) Prove, using mathematical induction, that

$$I_n = n\pi.$$ [8]

(*c*) Evaluate

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin^2 5\theta}{\sin^2 \theta}\right) d\theta.$$ [4]